

The University of Chicago

Founded by JOHN D. ROCKEFELLER

THE CROSS-RATIO GROUP OF 120 QUADRATIC CREMONA
TRANSFORMATIONS OF THE PLANE

A DISSERTATION

SUBMITTED TO THE FACULTIES OF THE GRADUATE SCHOOLS OF ARTS, LITERATURE
AND SCIENCE, IN CANDIDACY FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

[DEPARTMENT OF MATHEMATICS]

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HERBERT ELLSWORTH SLAUGHT

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*The Cross-Ratio Group of 120 Quadratic Cremona Transformations of the Plane.**

Part First: Geometric Representation.†

BY HERBERT ELLSWORTH SLAUGHT.

§1.—INTRODUCTION.

1. The group of Cremona transformations, whose geometric representation is the subject of the following paper, is a special case, $n = 5$, of the general cross-ratio group of order $n!$.‡ If these cross-ratio transformations be expressed in homogeneous point coordinates, then each takes the form

$$z'_1 : z'_2 : z'_3 = \phi(z_1, z_2, z_3) : \chi(z_1, z_2, z_3) : \psi(z_1, z_2, z_3).$$

2. The transformations of this system possess the following properties:

(a). They are all algebraic of order

$$\mu \leq 2.$$

(b). They are birational, those of order $\mu = 2$ having $2^2 - 1$ fundamental points at which the functions ϕ, χ, ψ vanish simultaneously.

(c). If S_i, S_j, S_k are the substitutions on the five indices with which the transformations A_i, A_j, A_k are respectively associated, and if

$$S_i S_j = S_k,$$

* The study of this group was undertaken as a dissertation at the University of Chicago at the suggestion and under the direction of Professor Moore. A brief abstract of the chief results was printed in *Science*, July 29, 1898.

† In Part Second the complete form-system of invariants of the group is discussed.

‡ E. H. Moore, "The Cross-Ratio group of $n!$ Cremona Transformations of Order $n - 3$ in Flat Space of $n - 3$ Dimensions" (*American Journal of Mathematics*, vol. XXII, pp. 336-342, 1900).

when compounded from left to right, then

$$A_i A_j = A_k,$$

when compounded from right to left.

3. This system of transformations forms a group distinguished by the following characteristics:

(a). It is holodrically isomorphic with the symmetric substitution group on five indices.

(b). It is a group of *quadratic Cremona transformations of the plane into itself*.

(c). It has four fundamental points, some three of which belong to every quadratic transformation in the group. See Art. 13.

(d). It is abstractly identical with, and under collineation equivalent to, the Cremona group of order 120 noted by Autonne* and Kantor;† but it is distinguished by the fact that its operators are all *cross-ratio transformations*.‡

4. In studying the geometric representation of this group, the following generational transformations are chosen. Corresponding to the substitutions

$$(34), (23)(45), (45), (15)(34) \text{ and } (12)$$

on the indices 1 5, we find the transformations respectively,

$$\begin{aligned} K; & z'_1 : z'_2 : z'_3 = z_3 : z_2 : z_1, \\ L; & z'_1 : z'_2 : z'_3 = z_3 - z_2 : z_3 - z_1 : z_3, \\ M; & z'_1 : z'_2 : z'_3 = z_2 : z_1 : z_3, \\ T; & z'_1 : z'_2 : z'_3 = z_3(z_1 - z_2) : (z_1 - z_2)(z_3 - z_2) : z_1(z_3 - z_2), \\ T'; & z'_1 : z'_2 : z'_3 = z_2 z_3 : z_1 z_3 : z_1 z_2. \end{aligned}$$

5. Of these, K , L and M , are collineations and T and T' are quadratic transformations. It is easily seen that all transformations corresponding to permutations on the indices 1 5 which leave 1 fixed are linear, and that all others are quadratic. Hence the 24 linear transformations by themselves form a subgroup $G_{24}^{(1)}$, which, it will be found, may be generated by K , L , M (Art. 7), and then either T or T' will extend $G_{24}^{(1)}$ to the main group G_{120} .

* Journal de Mathématiques, series 4, vol. I, 1885, p. 435.

† "Theorie der endlichen Gruppen von eindeutigen Transformationen in der Ebene," p. 105.

‡ E. H. Moore, American Journal of Mathematics, vol. XXII, p. 340, 1900.

§2.—Geometric Representation of $G_{24}^{(1)}$.

6. The Klein's* linear homogeneous substitution group G_{41} has been exhibited geometrically† in connection with the complete quadrangle (including the diagonals) whose vertices are‡

$$\left. \begin{aligned} O_1; \quad x_1 : x_2 : x_3 &= 1 : 1 : 1, \\ O_2; \quad x_1 : x_2 : x_3 &= 1 : -1 : 1, \\ O_3; \quad x_1 : x_2 : x_3 &= 1 : 1 : -1, \\ O_4; \quad x_1 : x_2 : x_3 &= -1 : 1 : 1, \end{aligned} \right\} \quad (1)$$

where the boundaries of the fundamental region are defined by‡

$$\left. \begin{aligned} x_3 &\geq 0, \\ x_1 - x_2 &\leq 0, \\ x_1 - x_3 &\geq 0. \end{aligned} \right\} \quad (2)$$

And the generators of the group are

$$\left. \begin{aligned} K'; \quad x'_1 : x'_2 : x'_3 &= x_3 : x_2 : x_1, \\ L'; \quad x'_1 : x'_2 : x'_3 &= x_1 : x_2 : -x_3, \\ M'; \quad x'_1 : x'_2 : x'_3 &= x_2 : x_1 : x_3. \end{aligned} \right\} \quad (3)$$

7. The linear subgroup $G_{24}^{(1)}$ possesses properties similar to those shown by Professor Moore for G_{41} . As a substitution group, it permutes among themselves the four indices 2, 3, 4, 5.

Geometrically, it permutes among themselves certain four points of the plane which may be found by considering the transformations K, L, M set up in Art. 4.

K is a projective reflection on the axis

$$z_1 - z_3 = 0$$

from the center

$$z_1 : z_2 : z_3 = -1 : 0 : 1.$$

L and M are ordinary reflections on the axes respectively,

$$\begin{aligned} z_1 + z_2 - z_3 &= 0, \\ z_1 - z_2 &= 0. \end{aligned}$$

* Klein, "Ueber eine geometrische Repräsentation der Resolventen algebraischer Gleichungen" (Mathematische Annalen, vol. IV, pp. 346-358, 1871).

† E. H. Moore, "Concerning Klein's Groups of $(n+1)!$ n -ary Collineations" (American Journal of Mathematics, vol. XXII, pp. 336-342, 1900).

‡ The fundamental region and the generators are here chosen in a way suitable for convenient use in this paper.

The plot of these lines [Fig. I]* shows that K, L, M permute the 4 points:

$$\left. \begin{aligned} Q_2; \quad z_1 : z_2 : z_3 &= 1 : 1 : 1, \\ Q_3; \quad z_1 : z_2 : z_3 &= 0 : 0 : 1, \\ Q_4; \quad z_1 : z_2 : z_3 &= 1 : 0 : 0, \\ Q_5; \quad z_1 : z_2 : z_3 &= 0 : 1 : 0. \end{aligned} \right\} \quad (4)$$

Now, a transformation can be found which throws

$$\begin{array}{cccc} O_1 & O_2 & O_3 & O_4 \\ Q_2 & Q_5 & Q_3 & Q_4 \end{array}$$

to

in the order indicated, namely,

$$\left. \begin{aligned} \text{direct:} \quad x_1 : x_2 : x_3 &= -z_1 + z_2 + z_3 : z_1 - z_2 + z_3 : z_1 + z_2 - z_3, \\ \text{inverse:} \quad z_1 : z_2 : z_3 &= x_2 + x_3 : x_3 + x_1 : x_1 + x_2. \end{aligned} \right\} \quad (5)$$

8. Furthermore, this transformation also throws the generators

$$\begin{array}{ccc} K' & L' & M' \\ K & L & M. \end{array}$$

to

Hence, it follows at once that K, L, M are the *generators* of $G_{24}^{(1)}$, and that Q_2, Q_3, Q_4, Q_5 are permuted among themselves by *every* transformation of the linear subgroup.

Because of the projective relation thus established between G_{41} and $G_{24}^{(1)}$, the complete quadrangle [with its diagonal lines] whose vertices are Q_2, Q_3, Q_4, Q_5 is the geometric representation for the latter group. See Fig. II.

The fundamental region† for the new configuration is derived from that defined in (2) by the transformation (5). It is defined‡ by

$$\left. \begin{aligned} z_1 - z_2 &\geq 0, \\ z_1 - z_3 &\leq 0, \\ z_1 + z_2 - z_3 &\geq 0. \end{aligned} \right\} \quad (6)$$

The new generators are, as they should be, edge-operators with reference to this region.

*For convenience, Fig's. I, VII, VIII, IX are placed together on the last page of plates.

†An independent determination of the fundamental region (6) is given in Arts. 51 to 57, where the transformation names of all the regions are derived, and certain generational relations are discovered from the configuration itself.

‡ Where z_1, z_2, z_3 are all + above $z_2 = 0$ and to the right of $z_1 = 0$.

The plane is divided into 24 regions which are permuted among themselves by every transformation of the group. That is, if we call this configuration Π_1 ,

$$G_{24}^{(1)} \Pi_1 = \Pi_1. \quad (7)$$

§3.—*The Subgroups Conjugate with $G_{24}^{(1)}$.*

9. The general theory* of Cremona transformations may be illustrated for the quadratic operators of G_{120} by considering the product $MT \sim (1534)^\dagger$ [Art. 4],

$$\left. \begin{aligned} z'_1 : z'_2 : z'_3 &= (z_1 - z_2)(z_3 - z_2) : z_3(z_1 - z_2) : z_1(z_3 - z_2), \\ \text{of which the inverse is} \\ z_1 : z_2 : z_3 &= z'_3(z'_2 - z'_1) : (z'_2 - z'_1)(z'_3 - z'_1) : z'_2(z'_3 - z'_1). \end{aligned} \right\} \quad (1)$$

In the z -plane, the fundamental points are

$$\left. \begin{aligned} Q_2; \quad z_1 : z_2 : z_3 &= 1 : 1 : 1, \\ Q_3; \quad z_1 : z_2 : z_3 &= 0 : 0 : 1, \\ Q_4; \quad z_1 : z_2 : z_3 &= 1 : 0 : 0, \end{aligned} \right\} \quad (2)$$

and the fundamental lines are

$$\left. \begin{aligned} Q_2 Q_3; \quad z_1 - z_2 &= 0, \\ Q_2 Q_4; \quad z_2 - z_3 &= 0, \\ Q_3 Q_4; \quad z_2 &= 0, \end{aligned} \right\} \quad (3)$$

while in the z' -plane the fundamental points are

$$\left. \begin{aligned} Q'_2; \quad z'_1 : z'_2 : z'_3 &= 1 : 1 : 1, \\ Q'_3; \quad z'_1 : z'_2 : z'_3 &= 0 : 0 : 1, \\ Q'_5; \quad z'_1 : z'_2 : z'_3 &= 0 : 1 : 0, \end{aligned} \right\} \quad (4)$$

and the fundamental lines are

$$\left. \begin{aligned} Q'_2 Q'_3; \quad z'_1 - z'_2 &= 0, \\ Q'_2 Q'_5; \quad z'_1 - z'_3 &= 0, \\ Q'_3 Q'_5; \quad z'_1 &= 0, \end{aligned} \right\} \quad (5)$$

*Clebsch, "Vorlesungen über Geometrie," vol I, pp. 474-496.

†Read, "The Operator MT Corresponding to the Substitution (1534)."

10. Evidently any operator whose square is identity, has the *same* fundamental points and lines in *both* planes (considered as superposed), though they are not necessarily associated in the same order in the two planes.

Indeed, this property is true of any one of the set of 6 transformations [Art. 19]

$$D_{1i}^{-1} \sim \{jkl\} \text{ all } (1i)^*, [i \neq j, \neq k, \neq l, = 2, 3 \dots 5],$$

In particular, any one of the set D_{12}^{-1} has the *coordinate vertices and sides* in some order as its fundamental points and lines in each plane.

Of these the quadratic inversion [Art. 4],

$$T' \sim (12); \quad z'_1 : z'_2 : z'_3 = z_2 z_3 : z_1 z_3 : z_1 z_2$$

is the simplest, having always a vertex associated with its *opposite* side in the coordinate triangle.

11. Consider the transformation under MT from the z -plane to the z' -plane :

(a) A straight line (in general)

$$\alpha z_1 + \beta z_2 + \gamma z_3 = 0$$

goes into a *non-degenerate* conic

$$\alpha z'_3 (z'_2 - z'_1) + \beta (z'_2 - z'_1) (z'_3 - z'_1) + \gamma z'_2 (z'_3 - z'_1) = 0,$$

passing through the three points (4).

(b) A line through one of the points (2)

$$\alpha z_2 + \beta (z_1 - z_2) = 0$$

goes into a *degenerate* conic consisting of another line through *some one of the points* (4) and its *opposite fundamental side*,

$$\begin{aligned} z'_1 - z'_2 &= 0, \\ \alpha z'_3 - (\alpha - \beta) z'_1 &= 0. \end{aligned}$$

In particular, a *side* of the quadrangle Π_1 through *one* fundamental point goes into a degenerate conic consisting of *two sides*, thus :

$$z_1 - z_3 = 0$$

* I use the notation of Cayley, Quarterly Journal, vol. 25, p. 73, except that I use the parentheses to indicate substitutions, and hence, indicate groups by $\{ \}$, so that the above form means the group $\{jkl\}$ all multiplied by the substitution $(1i)$ while $\{jkl\}$ all $\{1i\}$ means the same group multiplied by the group 1, $(1i)$.

goes into

$$z'_1 = 0 \text{ and } z'_2 - z'_3 = 0.$$

(c) A *fundamental* line, joining two of the points (2),

$$z_1 - z_2 = 0$$

goes into a degenerate conic consisting of two of the fundamental lines (5), namely

$$z'_1 = 0$$

and

$$z'_1 - z'_2 = 0.$$

See also (f) below for the correspondence of the points on such a line.

(d) An *ordinary* point, defined by the intersection of two (general) lines, goes into the fourth (movable) point of intersection of the two corresponding conics, the other three intersections being the fundamental points in the z' -plane.

(e) One of the fundamental points (3) corresponds to one of the fundamental sides (5) in the following sense: to each "*direction*" in which a movable point may approach the given fundamental point in the z -plane, there corresponds a definite point situated on a certain one of the fundamental lines in the z' -plane. Any such "*direction*" is determined by a *specific tangent in the fundamental point* and the corresponding point in the z' -plane is given by (b) above. For example, to the direction given by the tangent

$$\alpha z_2 + \beta (z_1 - z_2) = 0 \tag{6}$$

in the fundamental point

$$z_1 : z_2 : z_3 = 0 : 0 : 1$$

there corresponds the point in the z' -plane given by the intersection of the line

$$\alpha z'_3 - (\alpha - \beta) z'_1 = 0$$

with the fundamental side

$$z'_1 - z'_2 = 0. \tag{7}$$

Thus, when the tangent (6) varies through all directions, the fundamental side (7) becomes the locus of the corresponding points.

(f) A *fundamental point must, therefore, be regarded as a "pencil of directions,"* and this makes clear how a fundamental line in the z -plane goes, by (c), into two fundamental lines in the z' -plane. For to each point on such a line, there corresponds, by (e), a specific direction in a certain pencil of the z' -plane. Among these points, however, are two fundamental points, to each of which corresponds,

by the infinity of directions through it, a certain fundamental line in the z' -plane. These lines are themselves, therefore, two direction-tangents in the above pencil, which is, then, determined by their intersection.

12. Since, under the transformation MT , the pencils of directions (2) go, in some order, into the ranges of points (5), the former are called *the critical points* and the latter *the critical lines of the transformation*. All other points are non-critical under the transformation MT .

Likewise (4) and (3) are the critical points and lines of the inverse transformation $(MT)^{-1}$. Evidently $(MT)^{-1}$ throws any line of the plane into a conic passing through the critical points of MT .

13. Since, for every quadratic transformation in G_{120} , the critical points are certain three ($n^2 - 1$) of the four vertices, and the critical lines are three of the six sides of the complete quadrangle Π_1 , previously found [Art. 8], the group is said to have four critical points and six critical lines.

Classification of the Quadratic Transformations, Arts. 14–16.

14. It has been seen that all linear transformations in G_{120} are associated with the substitutions which leave the index 1 fixed. The quadratic transformations may be divided into four sets of 24 each, according as the index 1 is thrown to 2, 3, 4 or 5 by the corresponding substitutions.

15. Lemma. *If s_{1i} is a transformation corresponding to any particular substitution throwing 1 to i , and if S_{1i} represents the whole set of such transformations, then*

$$S_{1i} = G_{24}^{(1)} s_{1i} \sim \{2345\} \text{ all } (1i), \quad [i = 2, 3, 4, 5].$$

For the set S_{1i} must be such that when the corresponding substitutions are combined with the one belonging to s_{1i}^{-1} , there will result all substitutions leaving 1 fixed; that is, the substitution group corresponding to $G_{24}^{(1)}$, thus:

$$S_{1i} s_{1i}^{-1} = G_{24}^{(1)}.$$

Hence, multiplying on the right by s_{1i} ,

$$S_{1i} = G_{24}^{(1)} s_{1i} \tag{8}$$

or

$$S_{1i}^{-1} = (G_{24}^{(1)} s_{1i})^{-1} = s_{1i}^{-1} G_{24}^{(1)}. \tag{9}$$

16. THEOREM. *The critical points are the same for all quadratic transformations belonging to the same set S_{1i} , ($i = 2, 3, 4, 5$).*

For if s_{1i}^{-1} , any particular transformation of the set S_{1i}^{-1} , be applied to the complete quadrangle Π_1 [Art. 8] a definite configuration will result in which every line of Π_1 will have become a conic passing through three fixed points, namely, the critical points of s_{1i} [Art. 12]. If the new figure be called Π_i , [$i = 2, 3, 4, 5$], then

$$s_{1i}^{-1} \Pi_1 = \Pi_i. \quad (10)$$

Any other transformation of the set S_{1i}^{-1} , when applied to Π_1 , will produce the same figure Π_i , for, using equation (9),

$$S_{1i}^{-1} \Pi_1 = s_{1i}^{-1} G_{24}^{(1)} \Pi_1.$$

By equations (7) of Art. 8 and (10) above,

$$s_{1i}^{-1} G_{24}^{(1)} \Pi_1 = s_{1i}^{-1} \Pi_1 = \Pi_i.$$

Hence,

$$S_{1i}^{-1} \Pi_1 = \Pi_i. \quad (11)$$

Since each transformation of the set S_{1i}^{-1} throws the lines of Π_1 into conics intersecting in the same fixed points in Π_i , therefore, all the transformations in the set S_{1i} have the same critical points [Art. 12].

Configurations for the Subgroups $G_{24}^{(i)}$, Arts. 17-24.

17. If $G_{24}^{(1)}$ be transformed by any one of the set of transformations S_{1i} , the new subgroup is the same as that produced by any other transformer of the set, for if

$$s_{1i}^{-1} G_{24}^{(1)} s_{1i} = G_{24}^{(i)},$$

then, by use of (8) and (9),

$$S_{1i}^{-1} G_{24}^{(1)} S_{1i} = G_{24}^{(i)}. \quad (12)$$

This $G_{24}^{(i)}$ corresponds to the substitution group which leaves the index i fixed and permutes among themselves $1, j, k, l$, [$i \neq j, \neq k, \neq l, = 2, 3, 4, 5$].

To $G_{24}^{(i)}$ belongs a configuration related to Π_1 in the following manner:

Let g and g' be any pair of corresponding transformations in $G_{24}^{(1)}$ and $G_{24}^{(i)}$ respectively, so that

$$s_{1i}^{-1} g s_{1i} = g'.$$

Suppose P to be any point of the plane and P' its conjugate under g' , then, by (13),

$$s_{1i}^{-1} g s_{1i} P = P'. \quad (14)$$

If, now, $s_{1i} P = P_1$ and $gP_1 = P_2$, then must

$$s_{1i}^{-1} P_2 = P'.$$

Thus P_1 and P_2 , a pair of conjugate points under g , are thrown by s_{1i}^{-1} to P and P' respectively, a pair of conjugate points under g' .

Since (13) holds for every g and for every s_{1i}^{-1} , therefore, any transformation of the set S_{1i}^{-1} , operating on a pair of conjugate points under $G_{24}^{(1)}$, gives a corresponding pair of conjugate points under $G_{24}^{(i)}$. Therefore, the configuration connected with $G_{24}^{(i)}$ is derived by applying to Π_1 any transformation of the set S_{1i}^{-1} and the result is by (11) Π_i , ($i = 2 \dots 5$).

18. THEOREM. *The configuration Π_i is thrown into itself by all transformations of $G_{24}^{(i)}$.*

For by (12),

$$G_{24}^{(i)} \Pi_i = S_{1i}^{-1} G_{24}^{(1)} S_{1i} \Pi_i$$

and by (11),

$$\Pi_i = S_{1i}^{-1} \Pi_1.$$

Hence,

$$S_{1i} \Pi_i = S_{1i} S_{1i}^{-1} \Pi_1 = \Pi_1,$$

and, Art. 8,

$$G_{24}^{(1)} S_{1i} \Pi_i = G_{24}^{(1)} \Pi_1 = \Pi_1.$$

Then, by (11),

$$S_{1i}^{-1} G_{24}^{(1)} S_{1i} \Pi_i = S_{1i}^{-1} \Pi_1 = \Pi_i.$$

Therefore,

$$G_{24}^{(i)} \Pi_i = \Pi_i. \quad (15)$$

19. The notation for the 4 points in Π_1 has been so chosen [Art. 7] that the three critical points for the set of transformers S_{1i} are

$$Q_j, Q_k, Q_i. \quad [i \neq j, \neq k, \neq l, = 2, 3, 4, 5].$$

These are the three points through which pass all conics in Π_i .

Thus Q_i plays a particular rôle in the passage from Π_1 to Π_i , in that by any of the six transformations in the set D_{1i}^{-1} [Art. 10], it is unmoved, and the other three critical points become ranges of points on

$$Q_i Q_j, \quad Q_i Q_k, \quad Q_i Q_l,$$

while, by any other* transformation of the system S_{ii}^{-1} , some one of the three is carried to Q_i and the other two with Q_i become ranges of points on

$$Q_i Q_j, Q_i Q_k, Q_i Q_l.$$

20. It follows that in operating upon Π_1 by any one of the set of transformations

$$S_{ii}^{-1}, \quad (i = 2, 3, 4, 5),$$

the resulting configuration, Π_i , contains again the 4 *pencils* and 6 *sides* of Π_1 , while the diagonal lines become proper conics in Π_i . It is convenient for this purpose to choose the following special transformations:

$$\begin{aligned} s_{12} &\sim (12) & ; & \quad z'_1 : z'_2 : z'_3 = z_2 z_3 : z_1 z_3 : z_1 z_2, \\ s_{13} &\sim (132) & ; & \quad z'_1 : z'_2 : z'_3 = z_3 (z_3 - z_2) : z_3 (z_3 - z_1) : (z_3 - z_1)(z_3 - z_2), \\ s_{14} &\sim (1452) & ; & \quad z'_1 : z'_2 : z'_3 = z_1 (z_1 - z_3) : (z_1 - z_3)(z_1 - z_2) : z_1 (z_1 - z_2), \\ s_{15} &\sim (152) & ; & \quad z'_1 : z'_2 : z'_3 = z_2 (z_2 - z_3) : (z_2 - z_3)(z_2 - z_1) : z_2 (z_2 - z_1). \end{aligned}$$

The generators of the subgroups $G_{24}^{(i)}$ are found by transforming K, L, M through $s_{12}, s_{13}, s_{14}, s_{15}$ respectively. The three conics of Π_i are given by operating upon the diagonal lines of Π_1 by $s_{12}^{-1}, s_{13}^{-1}, s_{14}^{-1}, s_{15}^{-1}$ respectively. The boundaries of the fundamental regions of Π_i are shown by operating upon those of Π_1 by $s_{12}^{-1}, s_{13}^{-1}, s_{14}^{-1}, s_{15}^{-1}$ respectively.

The configurations, $\Pi_2 \dots \Pi_5$, are shown in the figures III, IV, V, VI.

§4.—THE CONFIGURATION FOR G_{120} .

Algebraic Study of the Configuration, Arts. 21–27.

21. If, now, the figures Π_i be superposed upon the complete quadrangle Π_1 , the resulting configuration Π admits the following algebraic verification as to the intersections of the twelve conics with the sides and diagonals of Π_1 .

22. Since [Art. 19] the three conics of Π_i each pass through

$$Q_j, Q_k, Q_l, \quad [i \neq j, \neq k, \neq l, = 2, 3, 4, 5],$$

but none of them through the fourth vertex, therefore, through each vertex pass
3. 3 = 9 conics.

23. Since [Art. 20] the 6 sides and 4 pencils of Π_1 are reproduced in Π_i , while the 3 diagonals in each case become proper conics, the intersections of Π

* The set D_{ii}^{-1} is included in the set S_{ii}^{-1} . See Art 15.

will evidently be of three kinds only, namely, (a) *conics with the sides*, (b) *conics with the diagonals*, (c) *conics with conics*.

24. Each side, in addition to the 2 vertices lying on it, through each of which pass 9 conics, has 2 other rational points through each of which passes one conic, thus :

On the sides	lie the rational points.
$z_1 = 0$	$0 : 1 : 2$ and $0 : 2 : 1$
$z_1 - z_3 = 0$	$1 : -1 : 1$ and $2 : 1 : 2$
$z_2 = 0$	$1 : 0 : 2$ and $2 : 0 : 1$
$z_2 - z_3 = 0$	$-1 : 1 : 1$ and $1 : 2 : 2$
$z_1 - z_2 = 0$	$1 : 1 : -1$ and $2 : 2 : 1$
$z_3 = 0$	$1 : 2 : 0$ and $2 : 1 : 0$

25. Each side has also one intersection with a diagonal line. It will be found in the succeeding grouptheoretic study that these belong to the same class as the intersections with the conics.

26. Each diagonal line has 4 points, through each of which pass 4 conics, one belonging to each of the figures Π_i .

Thus, on the lines

$$z_1 + z_2 - z_3 = 0, \quad z_1 - z_2 - z_3 = 0, \quad z_1 - z_2 + z_3 = 0$$

lie respectively the systems of 4 points :

$$\begin{array}{lll}
 -\lambda : 1 + \lambda : 1 & \lambda : -\lambda' : 1 & -\lambda : \lambda' : 1 \\
 -\lambda' : 1 + \lambda' : 1 & \lambda' : -\lambda : 1 & -\lambda' : \lambda : 1 \\
 1 + \lambda : -\lambda : 1 & " & 1 + \lambda : \lambda : 1 \\
 1 + \lambda' : \lambda' : 1 & 1 + \lambda' : \lambda' : 1 & \lambda' : 1 + \lambda' : 1
 \end{array}$$

wherein

$$\lambda = \frac{1 + \sqrt{5}}{2}, \quad \lambda' = \frac{1 - \sqrt{5}}{2}.$$

27. The only real intersections of the 12 conics among themselves are at the 4 vertices and at the above 12 irrational points on the 3 diagonals. But there is a set of 20 imaginary points, through each of which pass 3 conics. These will be discussed in the succeeding grouptheoretic study. The configuration Π is shown in Fig. X.

Grouptheoretic Study of the Configuration II, Arts. 28–46.

28. In order to discuss the systems of conjugate elements under the main group, a generating operator is needed which extends $G_{24}^{(1)}$ to G_{120} . Any transformation of G_{120} not contained in $G_{24}^{(1)}$ will serve, since the index of $G_{24}^{(1)}$ under G_{120} is a prime.

It will be convenient to choose for this purpose

$$T \sim (15)(34), \quad [\text{see Art. 4}],$$

since K , L and T are edge-operators for the region defined by

$$\begin{aligned} z_1 - z_3 &\leq 0, \\ z_1 + z_2 - z_3 &\geq 0, \\ z_1 z_2 - z_1 z_3 + z_2 z_3 &\leq 0, \end{aligned} \quad [\text{see Fig. XI}].$$

The old generator M is now expressible in terms of K , L , T , thus:

$$M = TKLKTLTK,$$

so that K , L , T generate the group G_{120} .

29. Under G_{120} any element—point, line or conic—will, in general, go into 120 conjugates. Certain elements may, however, go into fewer conjugates. Such a *special* element E is invariant under a subgroup $G\{E\}$, whose index under G_{120} indicates the number of elements in the conjugate system.

The element E may be designated by a notation n consisting of such a combination of the indices 1 . . . 5 as will characterize the corresponding substitution group $G'\{E\}$. This notation may be indicated on the group by $G^N\{E\}$.

Let S be any transformation in G_{120} . Then, if E is invariant under $G\{E\}$, $S^{-1}E$ will be invariant under $S^{-1}G\{E\}S$.

Let s be the substitution corresponding to S . Then, if n is the notation for E , ns will be the notation for $S^{-1}E$, wherein s acts as a substitution upon the indices in the cycles of the notation n . Likewise, the notation for SE is ns^{-1} .

The complete set of substitutions changing the notation of E to that of SE is given by $G'\{E\}s^{-1}$, and the transformations corresponding to the inverse of the substitutions in the set are the only ones in G_{120} which throw E to SE .

30. As an illustration of such a special element, the diagonal

$$z_1 + z_2 - z_3 = 0,$$

which is fixed by points under the transformation corresponding to the substitution (23)(45), is found to be invariant under the subgroup

$$G_8^{23,45} \sim \{2435\}_8.*$$

Its notation will then be 23.45, provided we agree

$$23.45 = 32.45 = 32.54 = 45.32 = 54.32 = 54.23, \text{ etc. ;}$$

that is, *the pairing is definitive, but not the sequence either of the pairs or the indices in a pair.*

The transformation

$$S \sim (134)$$

throws

$$z_1 + z_2 - z_3 = 0$$

to the conic

$$z_1 z_2 - 2z_2 z_3 + z_3^2 = 0.$$

This conic is fixed by points under the transformation $\sim (15)(24)$ and is invariant under the subgroup

$$G_8^{15,24} = S^{-1} G_8^{23,45} S \sim (134)^{-1} \{2435\}_8 (134).$$

Its notation is

$$15.24 = [23.45](134).\dagger$$

The complete set of transformations throwing the diagonal 23.45 to the conic 15.24 is given by the substitutions

$$[\{2435\}_8(134)]^{-1}.$$

31. Since the index of $G_8^{23,45}$ under G_{120} is 15, it would at once appear that the 3 diagonals and 12 conics form a conjugate system. This is shown in the following table, $\ddagger$ in which the generators K , L and T are applied successively to the diagonal, while its notation is transformed by the *inverse* of the correspond-

* Notation of Cayley. See foot-note to Art. 10.

$\dagger$ Article 29. This means operate on the cycles of the notation 23.45 with the substitution (134).

$\ddagger$ The result of operating on any diagonal or conic of the configuration by any transformation of G_{120} may be read at once from this table without algebraic computation.

ing substitutions (these being the same as the *direct*, since K, L, T are all of period 2).

Diagonals and Conics.		Notation.	$K \sim (34).$	$L \sim (23)(45).$	$T \sim (15)(34).$
$z_1 + z_2 - z_3$	$= 0$	23 . 45	24 . 35	23 . 45	13 . 24
$z_1 - z_2 - z_3$	$= 0$	24 . 35	23 . 45	24 . 35	14 . 23
$z_1 - z_2 + z_3$	$= 0$	25 . 34	25 . 34	25 . 34	12 . 34
$z_2^2 - z_1 z_3$	$= 0$	12 . 34	12 . 34	13 . 25	25 . 34
$z_1^2 - z_2 z_3$	$= 0$	12 . 35	12 . 45	13 . 24	14 . 25
$z_1 z_2 - z_3^2$	$= 0$	12 . 45	12 . 35	13 . 45	13 . 25
$z_2^2 - 2z_2 z_3 + z_1 z_3$	$= 0$	13 . 24	14 . 23	12 . 35	23 . 45
$z_1^2 - 2z_1 z_3 + z_2 z_3$	$= 0$	13 . 25	14 . 25	12 . 34	12 . 45
$z_1 z_2 - z_1 z_3 - z_2 z_3$	$= 0$	13 . 45	14 . 35	12 . 45	13 . 45
$z_2^2 - 2z_1 z_2 + z_1 z_3$	$= 0$	14 . 23	13 . 24	15 . 23	24 . 35
$z_1 z_2 - 2z_1 z_3 + z_3^2$	$= 0$	14 . 25	13 . 25	15 . 34	12 . 35
$z_1 z_2 + z_1 z_3 - z_2 z_3$	$= 0$	14 . 35	13 . 45	15 . 24	14 . 35
$z_1^2 - 2z_1 z_2 + z_2 z_3$	$= 0$	15 . 23	15 . 24	14 . 23	15 . 24
$z_1 z_2 - 2z_2 z_3 + z_3^2$	$= 0$	15 . 24	15 . 23	14 . 35	15 . 23
$z_1 z_2 - z_1 z_3 + z_2 z_3$	$= 0$	15 . 34	15 . 34	14 . 25	15 . 34

The 3 diagonals

$$2i . jk$$

$$(i, j, k, = 3, 4, 5)$$

belong to the *linear* subgroups

$$G_8^{2i . jk} \sim \{2jik\}_8.$$

The 12 conics $1i.jk$ ($i, j, k, = 2, 3, 4, 5$)
belong the *quadratic* subgroups

$$G_8^{1i.jk} \sim \{1jik\}_8.$$

32. Again, the side $z_1 = 0$

is found to be *fixed by points* under the transformation $\sim (24)$ and invariant under the subgroup

$$G_{12}^{24} \sim \{135\} \text{ all } \{24\}.*$$

Its notation will then be $24 = 42.$

Since the index of G_{12}^{24} is 10, *the 6 sides do not form a complete system*, but we have seen that under quadratic transformations a *pencil of directions* plays the same rôle as a *side of the quadrangle* Π_1 , so that the 4 pencils and 6 sides may form a conjugate system. In fact the transformation

$$S \sim (12)(34)$$

throws the side $z_1 = 0$

to the pencil at $z_1 : z_2 : z_3 = 0 : 0 : 1.$

This pencil is fixed by *directions* under the transformation $\sim (13)$ and is invariant under the subgroup

$$G_{12}^{13} = S^{-1} G_{12}^{24} S \sim [(12)(34)]^{-1} [\{135\} \text{ all } \{24\}] [(12)(34)].$$

The notation for the pencil is

$$13 = 24 [(12)(34)].\dagger$$

The complete set of operators throwing the side 24 into the pencil 13 is given by the substitutions

$$[\{135\} \text{ all } \{24\}] [(12)(34)].$$

33. The notation for the 10 elements and the proof that they form a closed system under the generators K, L, T , is as follows:

* See foot-note to Art. 10.

† This means operate upon the *notation* 24 with the *substitution* $(12)(34).$

Sides and Pencils.	Notation.	$K \sim (34).$	$L \sim (23)(45).$	$T \sim (15)(34).$
$1:1:1$	12	12	13	25
$0:0:1$	13	14	12	45
$1:0:0$	14	13	15	35
$0:1:0$	15	15	14	15
$z_3 = 0$	23	24	23	24
$z_1 = 0$	24	23	35	23
$z_2 = 0$	25	25	34	12
$z_1 - z_3 = 0$	34	34	25	34
$z_2 - z_3 = 0$	35	45	24	14
$z_1 - z_2 = 0$	45	35	45	13

The 6 sides,

$$ij = ji, \quad (i \neq j, = 2 \dots 5),$$

belong to the subgroups

$$G_{12}^{ij} \sim \{1kl\} \text{ all } \{ij\}, \quad (k, l \neq i, j, = 2 \dots 5).$$

The 4 pencils

$$1i = i1 \quad (i = 2 \dots 5),$$

belong to the subgroups

$$G_{12}^{1i} \sim \{jkl\} \text{ all } \{1i\}, \quad (j, k, l, \neq i, = 2 \dots 5).$$

34. Each line or conic in the above system is a *locus of fixed points* under the transformation corresponding to the substitution by which it is named. These are the *only such loci* under the operators of G_{120} , as will at once appear by the following complete list of fixed points for the different types of transformations:

35. *Period 2. Two Types.*(1). *Type (ij).*(a). When $i \neq j, = 2 \dots 5$, the side ij is a *locus of fixed points*.One point not in the locus ij is fixed, namely, the intersection of the diagonal $ij.kl$ with the side kl . $(k, l, \neq i, j, = 2 \dots 5).$ One direction is fixed in each of the pencils, $1k$ and $1l$, namely, that one whose tangent passes through the fixed point outside the locus ij .(b). When $i = 2 \dots 5$, the pencil $1i$ is a *centre of fixed directions* under the transformations $s_{1i} \sim (1i)$.

One direction is also fixed under these transformations in each of the pencils

$$1j, 1k, 1l, \quad (j, k, l, \neq i, = 2 \dots 5).$$

And one point is fixed in each of the sides passing through the vertex $1i$, namely,

$$jk, jl, kl.$$

(2). *Type (ij)(kl).*(a). When $i, j, k, l, = 2 \dots 5$, the diagonal $ij.kl$ is a *locus of fixed points* under the transformation $\sim (ij)(kl)$ and one point not in $ij.kl$ is fixed under the same transformation, namely, the intersection of the other two diagonals.(b). When $i = 1$ and $j, k, l = 2 \dots 5$, the conic $1j.kl$ is a *locus of fixed points* under the transformation $\sim (1j)(kl)$, and one direction also is fixed in the pencil $1i$, that one whose tangent is common to the conics

$$1k.jl \text{ and } 1l.jk.$$

36. *Period 3. One type, (ijk).*

$$(i, j, k = 1 \dots 5).$$

There is no locus of fixed points for this type, but there are two classes of *discrete fixed points*, namely,

(a). A pair of imaginary points through which pass the three conics [Art. 42]

$$ij.lm, ik.lm, jk.lm, \quad (i, j, k, l, m, = 1 \dots 5).$$

(b). A pair of imaginary points lying on the side (or imaginary directions in the pencil) (Art. 46)

$$lm, \quad (l, m = 1 \dots 5).$$

37. *Period 4. One type, $(ijkl)$.*

There is no locus of fixed points, but again two classes of *discrete* fixed points.

(a). Real points where two sides meet two diagonals. Such a *point* on a *side* is conjugate with a *direction* in a *pencil* whose tangent (one of the sides) is common to the two conics corresponding to the two diagonals. See Art. 43.

(b). *Imaginary points* which lie in pairs on the diagonals or conics. See Art. 45.

38. *Period 5. One type, $(ijklm)$.*

The only fixed points are real, discrete points whose coördinates involve the surd $\sqrt{5}$. These will be discussed in Art. 44.

39. *Period 6. One type, $(ij)(klm)$.*

Again, the only fixed points are imaginary, discrete points of the same form as class (b) under Period 3. See Art. 46.

40. We have thus found no loci of fixed points (or directions) except the 10 sides (and pencils) and 15 diagonals and conics which are exactly all the curves in the configuration II.

In order now to discover all the systems of conjugate points, with their respective notations, we take any known fixed point under a given type of transformation and find the subgroup which leaves it unmoved and determine its notation by means of the corresponding substitution group. Then we operate upon it with the generators of G_{120} , transforming the notation as in Arts. 31, 33, and continue the process till the system is closed.

The intersection points of II are classified as double, triple, quadruple and quintuple, according to the number of lines or conics passing through them.

Corresponding to a *double point lying on one of the sides* will be a *direction in one of the pencils, whose tangent belongs to one conic only, while to a quadruple point on one of the sides corresponds a direction in a pencil, whose tangent is common to two conics and coincides with a side passing through the pencil.*

Following is a complete enumeration of intersection points and their conjugate systems :

41. *Double Points.*

The point $\quad \quad \quad -1:0:1$

is invariant under transformations of Period 2 corresponding to

$$(34), (25) \text{ and } (25)(34),$$

and under no others except identity. It, therefore, belongs to the subgroup

$$G_4^{25,34} \sim \{(25)(34)\}.$$

The point $\quad \quad \quad 1:2:1$

is also invariant under the same subgroup. The notations for these two points are, then,

$$\overline{25}.34 \text{ and } \overline{34}.25,$$

where the *sequence of pairs is definitive but not the sequence of indices in a pair*. The first pair indicates the name of the side which is cut at the point by the diagonal 25.34 . The stroke distinguishes the notation from that of the diagonal $25.34 = 34.25$. [Art. 30.]

The complete list, consisting of 12 directions in the 4 pencils plus 18 points on the 6 sides is as follows:

"Direction" Tangents in the Pencils.			Notation.
$z_1 + z_3 = 0$	15		$\overline{12}.34$
$2z_1 - z_3 = 0$	15		$\overline{14}.23$
$z_1 - 2z_3 = 0$	15		$\overline{13}.24$
$z_2 + z_3 = 0$	14		$\overline{12}.35$
$2z_2 - z_3 = 0$	14		$\overline{15}.23$
$z_2 - 2z_3 = 0$	14		$\overline{13}.25$
$z_1 + z_2 = 0$	13		$\overline{12}.45$
$z_1 - 2z_2 = 0$	13		$\overline{15}.24$
$2z_1 - z_2 = 0$	13		$\overline{14}.25$
$z_1 + z_2 - 2z_3 = 0$	12		$\overline{13}.45$
$z_1 - 2z_2 + z_3 = 0$	12		$\overline{15}.34$
$-2z_1 + z_2 + z_3 = 0$	12		$\overline{14}.35$

Intersections on the	Sides.	Notation.
1: 2: 0	23	$\overline{23}.14$
2: 1: 0	23	$\overline{23}.15$
1:—1: 0	23	$\overline{23}.45$
0: 2: 1	24	$\overline{24}.13$
0: 1: 2	24	$\overline{24}.15$
0:—1: 1	24	$\overline{24}.35$
2: 0: 1	25	$\overline{25}.13$
1: 0: 2	25	$\overline{25}.14$
—1: 0: 1	25	$\overline{25}.34$
1:—1: 1	34	$\overline{34}.12$
2: 1: 2	34	$\overline{34}.15$
1: 2: 1	34	$\overline{34}.25$
—1: 1: 1	35	$\overline{35}.12$
1: 2: 2	35	$\overline{35}.14$
2: 1: 1	35	$\overline{35}.24$
1: 1:—1	45	$\overline{45}.12$
2: 2: 1	45	$\overline{45}.13$
1: 1: 2	45	$\overline{45}.23$

These points or directions, which lie by threes on the 6 sides and 4 pencils, are fixed by pairs under the 15 *distinct* conjugate subgroups of order 4. That is, the points

$$\overline{ij}.kl \text{ and } \overline{kl}.ij$$

belong to the groups $G_4^{ij,kl} \sim \{(ij)(kl)\}$. $[i, j, k, l, = 1 \dots 5]$.

42. Triple points.

The two imaginary points*

$$1 - \omega : 1 - \omega^2 : 1 \text{ and } 1 - \omega^2 : 1 - \omega : 1$$

are fixed under the transformation $\sim (254)$, and are invariant under the sub-

* ω and ω^2 are imaginary cube roots of unity.

group

$$G_6^{13.254} \sim [\{254\} \text{ all } \{12\}] \text{ pos.}$$

These points may be named respectively

$$13.245 \text{ and } 13.254,$$

where

$$13.245 = 31.254 \neq 31.245$$

and

$$13.254 = 31.245 \neq 31.254.$$

That is, the meaning of the notation is unchanged by reversing the cyclic order of indices in *both parts simultaneously*, but *not in either part alone*.

The points of this system are as follows:

20 Triple Points.	Notation.
$\omega : \omega^2 : 1$	12.345
$\omega^2 : \omega : 1$	12.354
1 — $\omega : 1$ — $\omega^2 : 1$	13.245
1 — $\omega^2 : 1$ — $\omega : 1$	13.254
1 — $\omega^2 : — 3\omega^2 : 3$	14.235
1 — $\omega : — 3\omega : 3$	14.253
— $3\omega^2 : 1$ — $\omega^2 : 3$	15.234
— $3\omega : 1$ — $\omega : 3$	15.243
1 — $\omega^2 : 1$ — $\omega : 3$	23.145
1 — $\omega : 1$ — $\omega^2 : 3$	23.154
1 — $\omega : — \omega : 1$	24.135
1 — $\omega^2 : — \omega^2 : 1$	24.153
— $\omega : 1$ — $\omega : 1$	25.134
— $\omega^2 : 1$ — $\omega^2 : 1$	25.143
$\omega : — \omega^2 : 1$	34.125
$\omega^2 : — \omega : 1$	34.152
— $\omega^2 : \omega : 1$	35.124
— $\omega : \omega^2 : 1$	35.142
— $\omega : — \omega^2 : 1$	45.123
— $\omega^2 : — \omega : 1$	45.132

These points belong in pairs to the 10 distinct conjugate subgroups

$$G_6^{ij.klm} = [\{klm\} \text{ all } \{ij\}] \text{ pos.}$$

Namely, to $G_6^{ij.klm}$ belong the 2 points

$$ij.klm \text{ and } ij.kml, \quad [i, j, k, l, m, = 1 \dots 5].$$

43. *Quadruple points.*

The point $0 : 1 : 1$

is fixed under the transformations corresponding to

$$(24)(35) \text{ and } (2345),$$

and is invariant under the subgroup

$$G_8^{24.35} \sim \{2345\}_8.$$

And the direction in the pencil 15,

$$z_1 - z_3 = 0,$$

is fixed under the transformations

$$(15)(34) \text{ and } (1354),$$

and is invariant under the subgroup

$$G_8^{15.34} \sim \{1354\}_8.$$

We name this point and direction

$$\overline{24} . \overline{35} \text{ and } \overline{15} . \overline{34}$$

respectively, where the *pairing only* is definitive, the strokes serving to distinguish these from the conics $15 . 34$, $24 . 35$, and also from the double points

$$\overline{15} . 34, \overline{34} . 15, \overline{24} . 35, \overline{35} . 24.$$

The complete set is as follows :

" Direction " Tangents in the Pencils.		Notation.
$z_1 - z_3 = 0$	12	$\overline{12} . \overline{34}$
$z_2 - z_3 = 0$	12	$\overline{12} . \overline{35}$
$z_1 - z_2 = 0$	12	$\overline{12} . \overline{45}$
$z_1 = 0$	13	$\overline{13} . \overline{24}$
$z_2 = 0$	13	$\overline{13} . \overline{25}$
$z_1 - z_2 = 0$	13	$\overline{13} . \overline{45}$
$z_3 = 0$	14	$\overline{14} . \overline{23}$
$z_2 = 0$	14	$\overline{14} . \overline{25}$
$z_2 - z_3 = 0$	14	$\overline{14} . \overline{35}$
$z_3 = 0$	15	$\overline{15} . \overline{23}$
$z_1 = 0$	15	$\overline{15} . \overline{24}$
$z_1 - z_3 = 0$	15	$\overline{15} . \overline{34}$
$0 : 1 : 1$		$\overline{24} . \overline{35}$
$1 : 0 : 1$		$\overline{25} . \overline{34}$
$1 : 1 : 0$		$\overline{23} . \overline{45}$

The points or directions of this system, $\overline{ij} . \overline{kl}$, are invariant under the 15 distinct subgroups

$$G_8^{ij.kl} \sim \{ikjl\}_8, \quad (i, j, k, l = 1 \dots 5).$$

44. Quintuple points.

The points* $-\lambda : 1 + \lambda : 1$ and $-\lambda' : 1 + \lambda' : 1$

are fixed under the transformation of period 5 corresponding to

$$(12543),$$

$$*\lambda = \frac{1+\sqrt{5}}{2}, \quad \lambda' = \frac{1-\sqrt{5}}{2}.$$

and are invariant under the subgroup

$$G_{10}^{12543} \sim \{12543\}_{10} \equiv [\{12543\}_{20}] \text{ pos.}$$

These points may be named respectively

$$12543 \text{ and } 14235,$$

with the understanding that the meaning of the notation is unchanged so long as the same *direct* or *reverse cyclic order* is maintained, thus

$$12543 = 25431 = 54321 = \dots = 13452 = 34521 = \dots$$

$$14235 = 42351 = 23514 = \dots = 15324 = 53241 = \dots$$

This system is as follows :

Quintuple Points.	Notation.
$-\lambda : 1 + \lambda : 1$	12543
$-\lambda' : 1 + \lambda' : 1$	14235
$1 + \lambda : -\lambda : 1$	12453
$1 + \lambda' : -\lambda' : 1$	14325
$\lambda : -\lambda' : 1$	15423
$\lambda' : -\lambda : 1$	12534
$1 + \lambda : \lambda : 1$	15243
$1 + \lambda' : \lambda' : 1$	14532
$-\lambda : \lambda' : 1$	15342
$-\lambda' : \lambda : 1$	14523
$\lambda : 1 + \lambda : 1$	13524
$\lambda' : 1 + \lambda' : 1$	12345

These points are fixed by twos under the 6 distinct conjugate subgroups; thus the two points

$$1ijkl \text{ and } 1kilj$$

are fixed under the subgroup

$$G_{10}^{1ijkl} \sim [\{1ijkl\}_{20}] \text{ pos.,} \quad (i, j, k, l, = 2 \dots 5),$$

45. There remain for consideration two systems consisting of imaginary fixed points which are not intersection points in the configuration Π .

One such system belongs to transformations of Period 4. Thus, the points*

$$1 - i : -i : 1 \text{ and } 1 + i : i : 1$$

are fixed under the transformation $\sim (2345)$, and are invariant under the subgroup

$$G_4^{2345} \sim \{2345\} \text{ cyc.}$$

These points may be named respectively

$$2345 \text{ and } 2543,$$

with the understanding that the meaning of the notation is unchanged so long as the same direct cyclic order is maintained, thus :

$$2345 = 3452 = 4523 = 5234,$$

$$2543 = 5432 = 4325 = 3254.$$

This system is as follows :

30 Complex Points.			Notation.
$2 : 1 + i :$	1		1234
$2 : 1 - i :$	1		1432
$1 : 1 + i :$	2		1243
$1 : 1 - i :$	2		1342
$-1 : -i :$	1		1324
$-1 : +i :$	1		1423
$1 + i : 2 :$	1		1235
$1 - i : 2 :$	1		1532
$1 + i : 1 :$	2		1253
$1 - i : 1 :$	2		1352
$-i : -1 :$	1		1325
$+i : -1 :$	1		1523
$1 : 2 : 1 + i$			1245
$1 : 2 : 1 - i$			1542

* $i = \sqrt{-1}$.

30 Complex Points.	Notation.
$2 : 1 : 1 + i$	1254
$2 : 1 : 1 - i$	1452
$+i : -i : 1$	1425
$-i : +i : 1$	1524
$1 : 1 + i : 1 - i$	1345
$1 : 1 - i : 1 + i$	1543
$1 + i : 1 : 1 - i$	1354
$1 - i : 1 : 1 + i$	1453
$1 - i : 1 + i : 1$	1435
$1 + i : 1 - i : 1$	1534
$1 - i : -i : 1$	2345
$1 + i : +i : 1$	2543
$-i : 1 - i : 1$	2354
$+i : 1 + i : 1$	2453
$1 : -i : 1 - i$	2435
$1 : +i : 1 + i$	2534

These points lie in pairs on the 15 conics and diagonals, namely, the two points

$$jklm \text{ and } jmlk$$

lie on

$$jl.km.$$

They also belong in pairs to the 15 distinct conjugate subgroups

$$G_4^{ijklm} \sim \{jklm\} \text{ cyc.}, \quad (j, k, l, m, = 1 \dots 5).$$

46. Finally, the points

$$-\omega : 0 : 1 \text{ and } -\omega^2 : 0 : 1$$

are fixed under transformation of Periods 3 and 6, and belong to the subgroup

$$G_6^{134} \sim \{134\} \text{ cyc. } \{25\}.$$

And the directions

$$\omega z_1 + z_2 = 0 \text{ and } \omega^2 z_1 + z_2 = 0$$

are invariant under

$$G_6^{245} \sim \{245\} \text{ cyc. } \{13\}.$$

These points and directions may be named respectively

$$134, 143, 245, 254,$$

the *direct cyclic order only being definitive*.

This system is as follows :

"Direction" Tangents in the	Pencils.	Notation.
$\omega z_1 + z_2 + \omega^2 z_3 = 0$	12	345
$\omega^2 z_1 + z_2 + \omega z_3 = 0$	12	354
$\omega z_1 + z_2 = 0$	13	245
$\omega^2 z_1 + z_2 = 0$	13	254
$z_2 + \omega z_3 = 0$	14	235
$z_2 + \omega^2 z_3 = 0$	14	253
$z_1 + \omega z_3 = 0$	15	234
$z_1 + \omega^2 z_3 = 0$	15	243
Complex Points on the	Sides.	
$1 : -\omega^2 : 0$	23	145
$1 : -\omega : 0$	23	154
$0 : -\omega^2 : 1$	24	135
$0 : -\omega : 1$	24	153
$-\omega : 0 : 1$	25	134
$-\omega^2 : 0 : 1$	25	143
$1 : -\omega^2 : 1$	34	125
$1 : -\omega : 1$	34	152
$-\omega : 1 : 1$	35	124
$-\omega^2 : 1 : 1$	35	142
$-\omega : -\omega : 1$	45	123
$-\omega^2 : -\omega : 1$	45	132

These points or directions lie in pairs on the 10 sides and pencils, and belong in pairs to the 10 distinct conjugate subgroups; thus the points ijk and ikj belong to

$$G_6^{ijk} = G_6^{ikj} \sim \{ijk\} \text{ cyc. } \{lm\},$$

$$(i, j, k, l, m, = 1 \dots 5).$$

Geometric Study of the Configuration Π , Arts. 47–50.

47. In the preceding grouptheoretic study, we have found loci of two types:

- (a) The 6 + 4 sides and pencils ij .
- (b) The 3 + 12 diagonals and conics $ij.kl$;

and real intersection points of three types:

- (a) The 30 double points $\overline{ij}.kl$.
- (b) The 15 quadruple points $\overline{ij}.\overline{kl}$.
- (c) The 12 quintuple points $ijklm$, $[i, j, k, l, m, = 1 \dots 5]$.

48. These points are distributed in the following manner:

- (1) On the line of type $ij.kl$ are two points,

$$\overline{ij}.kl \text{ and } \overline{kl}.ij.$$

On the line (or pencil) of type ij are 3 points or directions,

$$\overline{ij}.kl, \overline{ij}.km, \overline{ij}.lm.$$

Hence we enumerate

$$2.15 = 3.10 = 30 \text{ double points.}$$

- (2) On the line $ij.kl$ are 2 points

$$\overline{ik}.j\overline{l} \text{ and } \overline{il}.j\overline{k},$$

and through such a point $\overline{ik}.j\overline{l}$ pass 2 curves

$$ij.kl \text{ and } il.kj.$$

On the line of type ij are 3 points,

$$\overline{ij}.\overline{kl}, \overline{ij}.\overline{km}, \overline{ij}.\overline{lm},$$

and through such a point $\overline{ij}.\overline{kl}$ pass 2 lines

$$ij \text{ and } kl,$$

thus giving

$$\frac{2.15}{2} = \frac{3.10}{2} = 15 \text{ quadruple points.}$$

- (3) On a curve of type $ij.kl$ are 4 points,

$$mijk, milkj, mkijl, mkjil,$$

and through such a point $miklj$ pass 5 curves,

$$ij.kl, mi.kj, mk.lj, ml.ik, mj.il,$$

thus enumerating

$$\frac{4 \cdot 15}{5} = 12 \text{ quintuple points.}$$

49. A line of type ij contains 6 intersection points, always in the cyclic order,

$$4, 2, 4, 2, 4, 2, \quad [\text{see Fig. X}],$$

where the numbers indicate the multiplicity of the points, thus :

(a) On the finite side 25 :

$$\overline{14.25}, \overline{25.34}, \overline{13.25}, \overline{25.14}, \overline{25.34}, \overline{25.13}.$$

(b) On the line at infinity 23, starting at $\overline{23.45}$, the intersection with the diagonal 23.45, and reading clockwise,

$$\overline{23.45}, \overline{15.23}, \overline{23.14}, \overline{23.45}, \overline{23.15}, \overline{14.23}.$$

(c) In a finite pencil 13, starting at $\overline{13.25}$, the intersection with the side 24, and reading counter-clockwise [see Fig. XI],

$$\overline{13.25}, \overline{13.24}, \overline{13.45}, \overline{13.25}, \overline{13.24}, \overline{13.45},$$

(d) In a pencil at infinity 14, starting at $\overline{14.23}$, where the parabolas 12.34 and 13.24 are tangent to the line at infinity 23, and reading counter-clockwise,

$$\overline{14.23}, \overline{14.25}, \overline{14.35}, \overline{14.23}, \overline{14.25}, \overline{14.35}.$$

50. A line of type $ij.kl$ has 8 intersection points always in the cyclic order,

$$2, 5, 4, 5, 2, 5, 4, 5,$$

the numbers indicating the multiplicity, thus :

(a) On a diagonal 23.45 :

$$\overline{23.45}, 12543, \overline{24.35}, 14325, \overline{45.23}, 14235, \overline{25.34}, 12453.$$

(b) On the hyperbola 15.23, starting in the pencil 15 and reading continuously along the curve

$$\overline{15.23}, 14523, \overline{12.35}, 15243, \overline{23.15}, 15342, \overline{13.25}, 14532.$$

(c) On an equilateral hyperbola 15.34 , starting in the pencil 14 ,

$$\overline{14.35}, 12543, \overline{15.34}, 12534, \overline{13.45}, 14235, \overline{34.15}, 15423.$$

(d) On a parabola 12.35 , starting at infinity and reading counter-clock-wise,

$$\overline{15.23}, 12543, \overline{35.12}, 12345, \overline{13.25}, 14235, \overline{12.35}, 13524.$$

It is to be noted here that $\overline{15.23}$ is a direction in the pencil 15 , whose tangent 23 is common to the two parabolas,

$$12.35 \text{ and } 13.25,$$

and thus the pencil 15 is said to be cut by

$$23, 12.35 \text{ and } 13.25,$$

forming the equivalent of a quadruple point. [Art. 40.]

Fundamental Region for G_{120} . Arts. 51–65.

51. If the fundamental region is known for any subgroup, that of the main group may be found by successively extending the subgroup and subdividing its region.

Let F_i represent the fundamental region for the subgroup G_i , then for

$$G_1 = \{1\}, \quad F_1 = \text{the whole plane.}$$

52. The extender

$$L \sim (23)(45)$$

leads to the subgroup

$$G_2 = \{L\} \sim \{23.45\},$$

with the relation

$$L_2 = 1.$$

F_2 is then defined* by

$$\begin{aligned} z_3 &\geq 0, \\ z_1 + z_2 - z_3 &\geq 0. \end{aligned} \quad [\text{See Fig. VII.}]$$

53. Again, the extender

$$M \sim (45)$$

leads to the subgroup

$$G_4 = \{L, M\} \sim \{(23)(45)\}$$

* With the understanding that z_1, z_2, z_3 are all + above $z_2 = 0$ and to the right of $z_1 = 0$.

with the relations

$$L^2 = M^2 (LM)^2 = 1.$$

The defining inequalities of F_4 are now

$$\begin{aligned} z_3 &> 0, \\ z_1 - z_2 &> 0, \\ z_1 + z_2 - z_3 &\geq 0. \end{aligned} \quad [\text{See Fig. VIII.}]$$

The generators L and M are edge-operators on the boundaries of F_4 , and since the repetition of these operators can introduce no new lines, F_4 is at once a *simple region*.

54. The extender $K \sim (34)$

leads to the subgroup

$$G_{24}^{(1)} = \{K, L, M\} \sim \{2345\} \text{ all}$$

with the relations [see Fig. II],

$$K^2 = L^2 = M^2 = (LM)^2 = (MK)^3 = LK^4 = 1.$$

Since the index of G_4 under $G_{24}^{(1)}$ is 6, it follows that F_4 should be partitioned into 6 simple regions by the generators of $G_{24}^{(1)}$. In fact, K, L, M are edge-operators on the lines

$$34, 23.45 \text{ and } 45.$$

It is easy to show that the repetition of these transformations gives no lines entering the region F_4 except

$$23, 34, 35 \text{ and } 23.45,$$

and these produce the six subdivisions defined as follows:

$$\begin{aligned} \text{(I).} \quad & z_1 + z_2 - z_3 \geq 0, \quad z_1 - z_2 \geq 0, \quad z_1 - z_3 \leq 0. \\ \text{(II).} \quad & z_1 - z_3 \geq 0, \quad z_2 - z_3 \geq 0, \quad z_1 - z_2 - z_3 \leq 0. \\ \text{(III).} \quad & z_1 - z_2 \geq 0, \quad z_2 - z_3 \geq 0, \quad z_1 - z_2 - z_3 \leq 0. \\ \text{(IV).} \quad & z_3 \geq 0, \quad z_2 - z_3 \geq 0, \quad z_1 - z_2 - z_3 \geq 0. \\ \text{(V).} \quad & z_2 \geq 0, \quad z_2 - z_3 \geq 0, \quad z_1 - z_2 - z_3 \geq 0. \\ \text{(VI).} \quad & z_3 \geq 0, \quad z_2 \leq 0, \quad z_1 + z_2 - z_3 \geq 0. \end{aligned}$$

56. By use of these inequalities and the three generators K, L, M , it may be readily shown:

(a) Each of these six regions is simple; that is, it is not subdivided by any line of Π_1 .

(b) Any point in the plane *outside* one of these regions, say (1), has *one* and *only one* conjugate point within that region.

(c) Any point *within* one of these regions has no other conjugate point within it.

We may then name region (1) F_{24} , in agreement with Art. 8, (6), and the others in order will be

$$K, KM, KLM, KL, KLK.$$

See Fig. IX.

Since each boundary of F_{24} is a fixed axis of reflection for one of the generators, it follows that the boundaries, including the corner points, count as a part of the region F_{24} , and hence of every conjugate region in Π_1 .

57. Applying the six extenders to G_4 , we get the rectangular table for $G_{24}^{(1)}$:

1	L	M	LM
K	LK	LM	LMK
KM	LKM	MKM	$LMKM$
KL	LKL	MKL	$LMKL$
KLK	$LKLK$	$MKLK$	$LMKLK$
KLM	$LKLM$	$MKLM$	$LMKLM$

These are the operators of $G_{24}^{(1)}$ as shown in Fig. II.

58. Finally, the extender

$$T \sim (15)(34)$$

leads to the main group

$$G_{120} \sim \{12345\} \text{ all.}$$

It is to be shown that F_{24} is divided into *five simple regions* by the lines and conics of the configuration Π , Fig. X.

To show what curves enter F_{24} .

- (a) No curve of Π_1 can cut F_{24} , since this is a fundamental region for $G_{24}^{(1)}$.
- (b) Hence, the only possibilities of partition within F_{24} are by the 12 conics.

Of these, it has been shown [§4] that none pass through the vertex $\overline{25.34}$, none through the vertex $\overline{45.23}$, 9 through the vertex 12, one through point $\overline{34.15}$, on the boundary 34, none through any point on the boundary 45, and 4 through the point 14235 on the boundary 23.45.

There are thus only three possible points of entrance into F_{24} ,

$$12, \overline{34.15} \text{ and } 14235.$$

It remains to show which of the conics through these points actually cut the boundary and enter the region. For this purpose it is convenient to use non-homogeneous coordinates, putting

$$\frac{z_1}{z_3} = \rho, \quad \frac{z_2}{z_3} = \sigma.$$

59. The interior of F_{24} is then defined by

$$\rho - 1 < 0, \tag{1}$$

$$\rho - \sigma > 0, \tag{2}$$

$$\rho + \sigma - 1 > 0. \tag{3}$$

$$\text{From (1) and (3),} \quad \sigma > 0. \tag{4}$$

$$\text{From (2) and (3),} \quad \rho > \frac{1}{2}. \tag{5}$$

Consider any one of the conics through 14235, say 13.24,

$$\sigma^2 - 2\sigma + \rho = 0, \tag{6}$$

$$\text{from which} \quad \rho = 2\sigma - \sigma^2. \tag{7}$$

$$\text{Substitute (7) in (1),} \quad 2\sigma - \sigma^2 - 1 < 0.$$

$$\text{Hence,} \quad (\sigma - 1)^2 > 0. \tag{8}$$

$$\text{Substitute (7) in (2).} \quad 2\sigma - \sigma^2 - \sigma > 0,$$

$$\text{or} \quad \sigma - \sigma^2 > 0.$$

$$\text{Hence,} \quad \sigma > 0. \tag{9}$$

which agrees with (4).

$$\text{Substitute (7) in (3),} \quad 2\sigma - \sigma^2 + \sigma - 1 > 0,$$

$$\text{or} \quad \sigma - (\sigma - 1)^2 > 0.$$

$$\text{Hence,} \quad \sigma > 0. \tag{10}$$

which agrees with (4).

Since, by (8), (9) and (10), the defining inequalities of F_{24} are consistent with (7), therefore, the conic 13. 24 enters the region F_{24} .

In like manner the other conics

$$12.35, 14.25 \text{ and } 15.24$$

may be shown to *cut* the boundary at 14235 and enter the region F_{24} .

60. Of these four conics, 15. 34 passes through the point $\overline{34}.15$, and hence must *cut* the boundary 34 at that point. The other three pass through the vertex 12, and hence must pass out of F_{24} at that point.

Moreover, *none of the remaining 9 conics through 12 can enter the region F_{24}* , since the only points of exit, $\overline{34}.15$ and 14235, have already their maximum number.

Hence, precisely 4 conics pass through the region F_{24} , and, therefore, *the partition into at least 5 parts is established.*

61. *To show that each of these divisions is a simple region*, we name and define them in order as follows, starting at the vertex $\overline{25}.34$ and passing along the boundary 23.45 toward the vertex $\overline{45}.23$.

$$1 \quad ; \quad \rho - 1 \leq 0, \quad \rho + \sigma - 1 \geq 0, \quad \rho\sigma - \rho + 1 \leq 0.$$

$$T \quad ; \quad \rho - 1 \leq 0, \quad \rho\sigma - \rho + \sigma \geq 0, \quad \sigma^2 - 2\sigma + \rho \geq 0.$$

$$TL \quad ; \quad \rho^2 - \sigma \geq 0, \quad \sigma^2 - 2\sigma + \rho \leq 0.$$

$$TLT \quad ; \quad \rho^2 - \sigma \leq 0, \quad \rho\sigma - 2\rho + 1 \geq 0.$$

$$TLTL; \quad \rho - \sigma \geq 0, \quad \rho + \sigma - 1 \geq 0, \quad \rho\sigma - 2\rho + 1 \geq 0.$$

The only possibility of subdivision in any one of these regions is by one of the 4 conics (just shown to enter F_{24}), which is *not* a *boundary* of the region in question. For instance, the region 1, whose interior is defined by

$$\rho - 1 < 0, \tag{1}$$

$$\rho + \sigma - 1 > 0, \tag{2}$$

$$\rho\sigma - \rho + \sigma > 0, \tag{3}$$

could be subdivided only by

$$12.35; \quad \rho^2 - \sigma = 0, \tag{4}$$

$$\text{or} \quad 13.24; \quad \sigma^2 - 2\sigma + \rho = 0, \tag{5}$$

or
$$14.25; \quad \rho\sigma - 2\rho + 1 = 0. \quad (6)$$

Substitute the value of σ from (4) in (2),

$$\rho^2 + \rho - 1 > 0. \quad (7)$$

Substitute the same in (3) and find

$$\rho^2 + \rho - 1 < 0. \quad (8)$$

As (7) and (8) are contradictory, 12.35 does not enter region 1.

Substitute the value of ρ from (5) in (3),

$$\sigma^2 - 3\sigma - 1 > 0. \quad (9)$$

Also in (2),
$$-\sigma^2 + 3\sigma - 1 > 0. \quad (10)$$

Add (9) and (10),
$$-2 > 0.$$

As this is impossible, 12.34 has no points with region 1.

Finally, substitute the value of ρ from (6) in (3) and find

$$\sigma^2 - 3\sigma + 1 > 0. \quad (11)$$

Also in (2),
$$-\sigma^2 + 3\sigma - 2 > 0. \quad (12)$$

Add (11) and (12),
$$-1 > 0.$$

Hence, 14.25 does not enter the region 1. Therefore, 1 is a simple region, and in the same manner each of the other four regions may be proved simple.

The names already given to these four regions correspond to the transformations by which they are respectively derived from the region 1.

62. These transformations

$$1, T, TL, TLT, TLTL,$$

are the proper extenders with which to form the rectangular table for G_{120} from the operators of $G_{24}^{(1)}$, for

(a) They are all different from those of $G_{24}^{(1)}$, since no two points in F_{24} can be conjugate under $G_{24}^{(1)}$.

(b) They are distinct from one another, since the supposition of equality between any two of them leads to a contradiction.

Such a table would contain initially the four generators

$$K, L, M, T,$$

from which, however, M may be eliminated by the relation

$$M = TKLKTLTK, \quad (1)$$

where T , L and K are *edge-operators* on the boundaries of region 1; that is, each generator has for its axis of reflection one of the boundary curves which it leaves fixed by points.

The table is not given in full, but the thirty subdivisions of F_4 are marked in Fig. XI, and the others (some of which are indicated), may be read at once by applying to these the transformations of

$$G_4 = \{L, M\} \sim \{(23)(45)\},$$

together with the relation (1).

63. Since F_{24} is shown to be partitioned into 5 simple regions, therefore, so is each of the 24 divisions of Π_1 .

Hence the configuration Π contains precisely 120 regions and the transformations of G_{120} throw region 1 to these 120 regions respectively, each bearing uniquely the name of the transformation by which it is derived from region 1.

Therefore, the conditions are fulfilled for a fundamental region

$$1 = F_{120},$$

in which the boundaries and vertices count as a part of the region.

(a) No two points of F_{120} are conjugate under any transformation of G_{120} , since every such operator throws each point of F_{120} to a *conjugate point in the new region whose name is the transformation in question*.

(b) Any point in the plane has *at least one* conjugate point in F_{120} ; for every point belongs to one of the 120 regions, and hence is thrown to a point in F_{120} by the inverse of the transformation naming that region.

(c) No point in the plane can be conjugate to *two points* of F_{120} , since these two points would then be conjugate to each other, contrary to (a).

64. F_{120} is a triangle, two of whose sides are of type $ij.kl$, and the third of type ij , while the vertices are double, quadruple, and quintuple points respectively, thus including all types of real points and lines in the configuration Π .

Hence, each of the 120 regions possesses the same characteristics. The *apparently* two-sided figures have a *pencil* as the third side of type $1i$, and *directions* in that pencil as the *double* and *quadruple* vertices. [See Art. 40.]

Thus, region TL has the boundaries

$$13.24, 12.35 \text{ and the pencil } 12$$

and it has the vertices

$$14235, \overline{12.34} \text{ and } \overline{12.35}.$$

65. Certain generational relations may now be read from the combination of transformations about the edges and vertices of the fundamental region.

(1) Since K , L and T are reflections on the three boundaries of F_{120} , then

$$K^2 = L^2 = T^2 = 1.$$

(2) KT is a rotation about a double point in either direction through 180° .

(3) KL is a clockwise rotation about a quadruple point through 90° .

(4) LT is a clockwise rotation about a quintuple point through 72° .

Hence, $(KT)^2 = (KL)^4 = (LT)^5 = 1$.

The abstract generational conditions also require other relations in addition to the above.*

THE UNIVERSITY OF CHICAGO, *July* 13, 1900.

* E. H. Moore, "Concerning the Abstract Groups of Order $k!$ and $\frac{1}{2}k!$ Holoedrally Isomorphic with the Symmetric and Alternating Substitution Groups on k Letters" (Proceedings of the London Mathematical Society), vol. XXVIII, No. 597, pp. 357-366. Also, "Concerning Klein's Group of $(n+1)!$ n -ary Collineations" (American Journal of Mathematics, vol. XXII, pp. 341-342, §10, 1900).

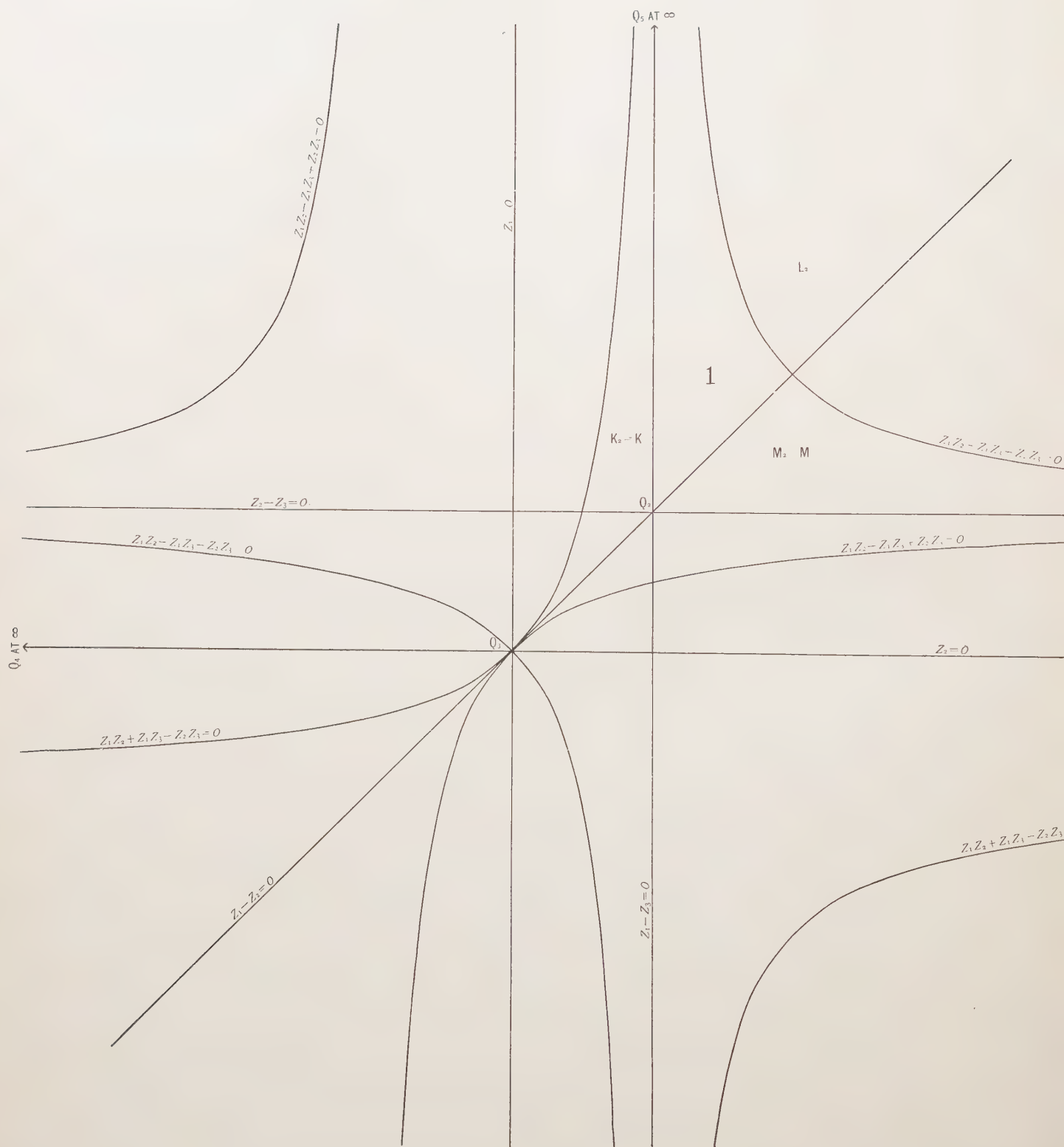


Fig. III.

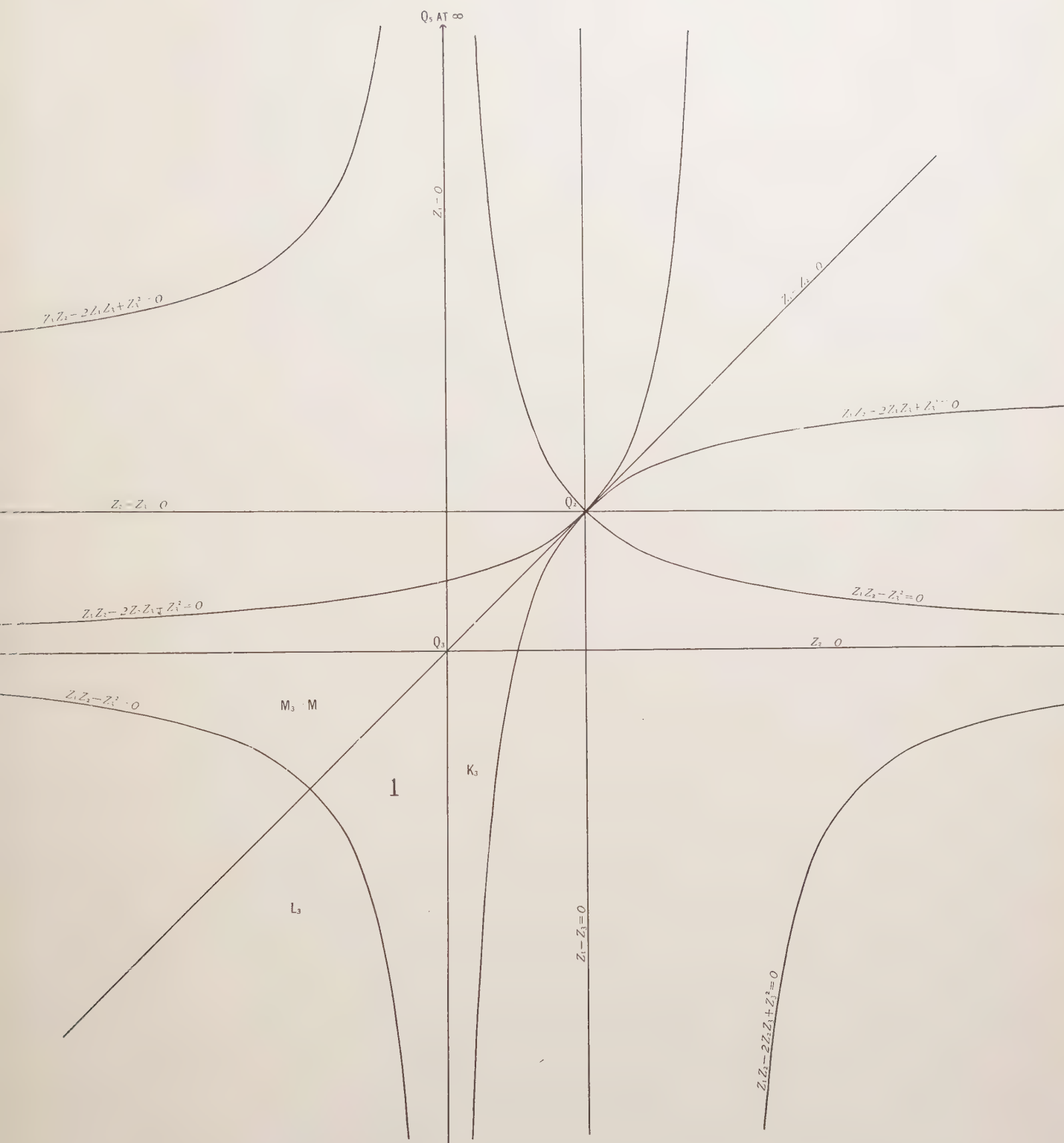


Fig. IV.

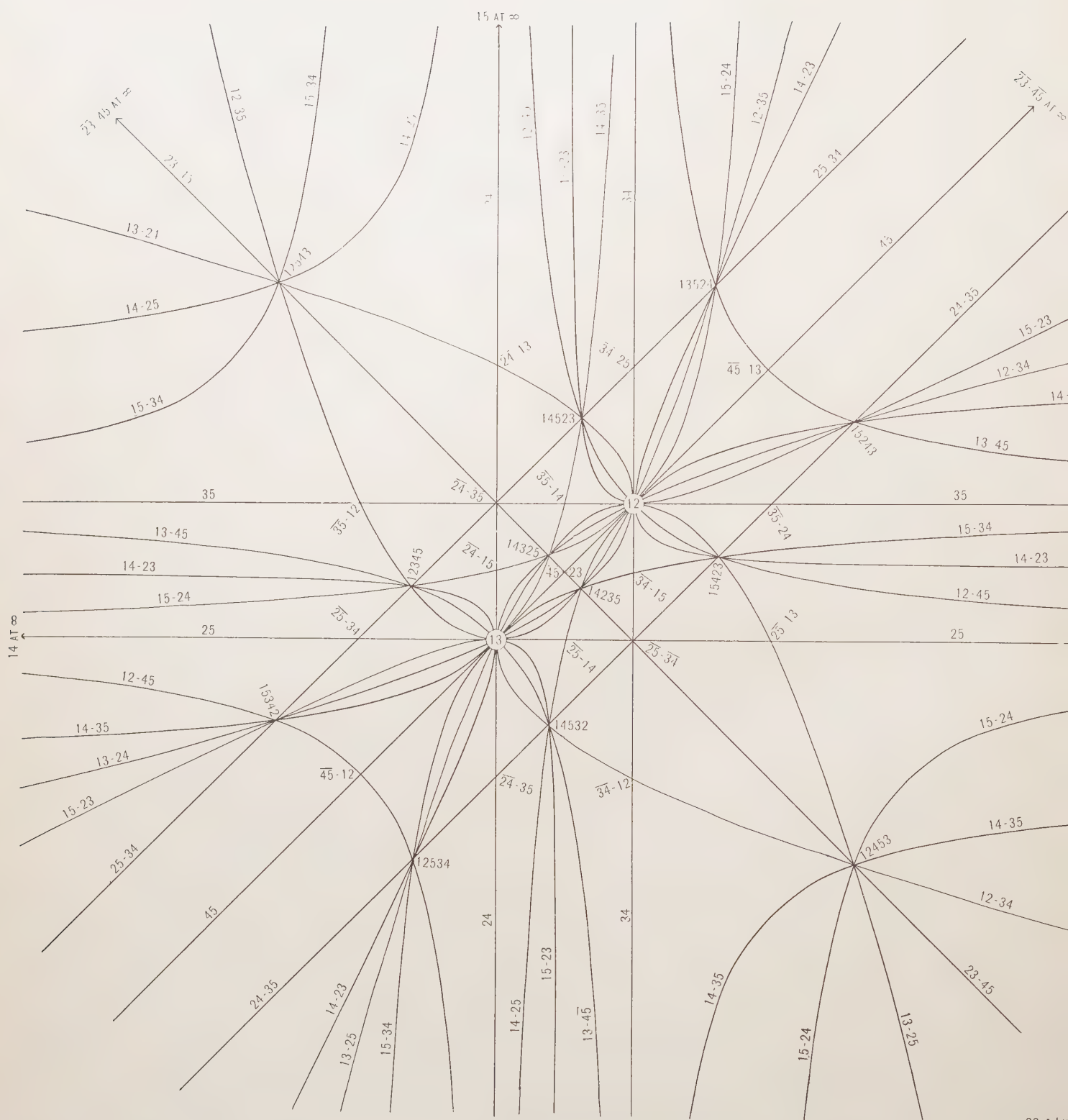
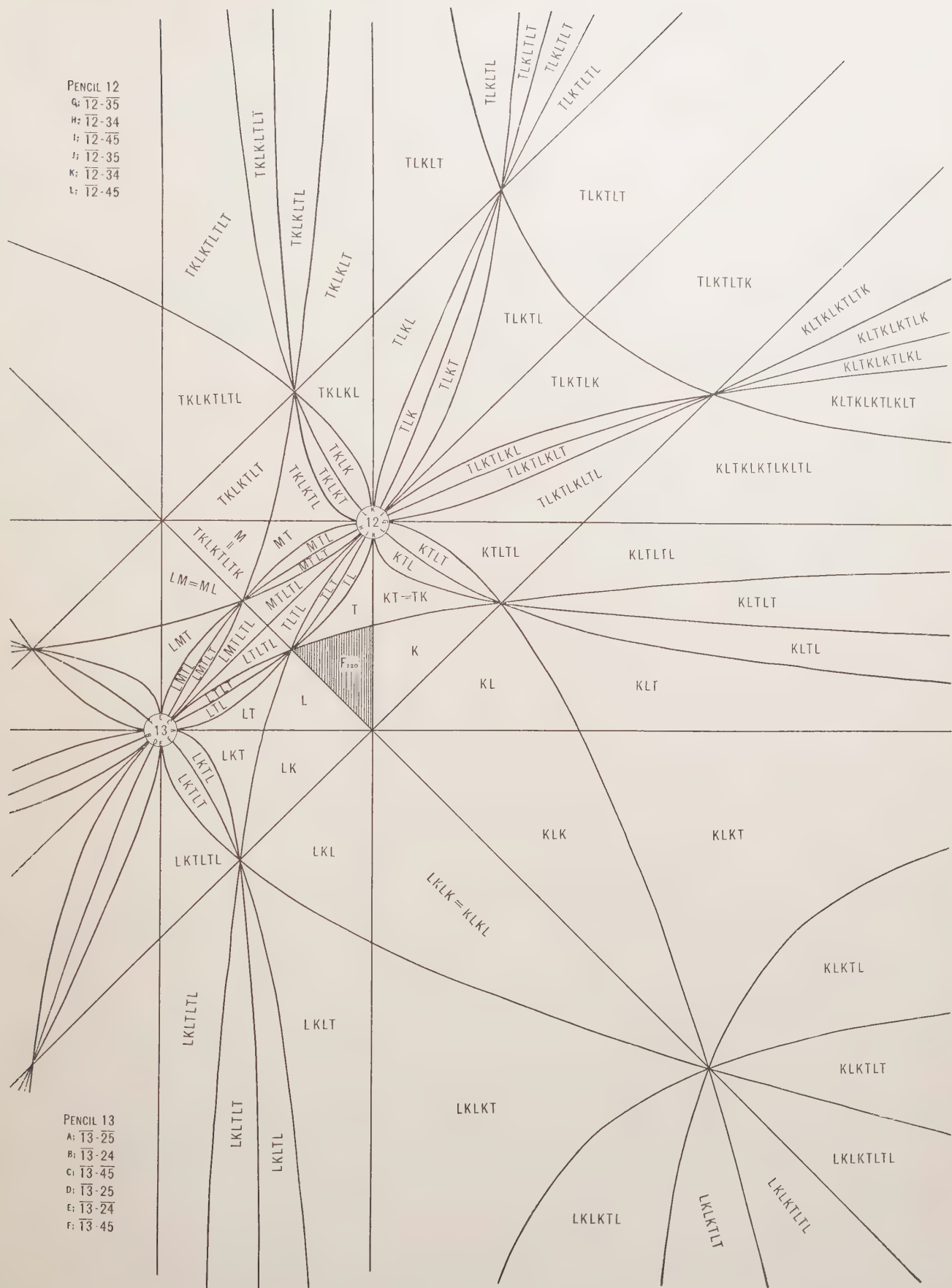


Fig. X.



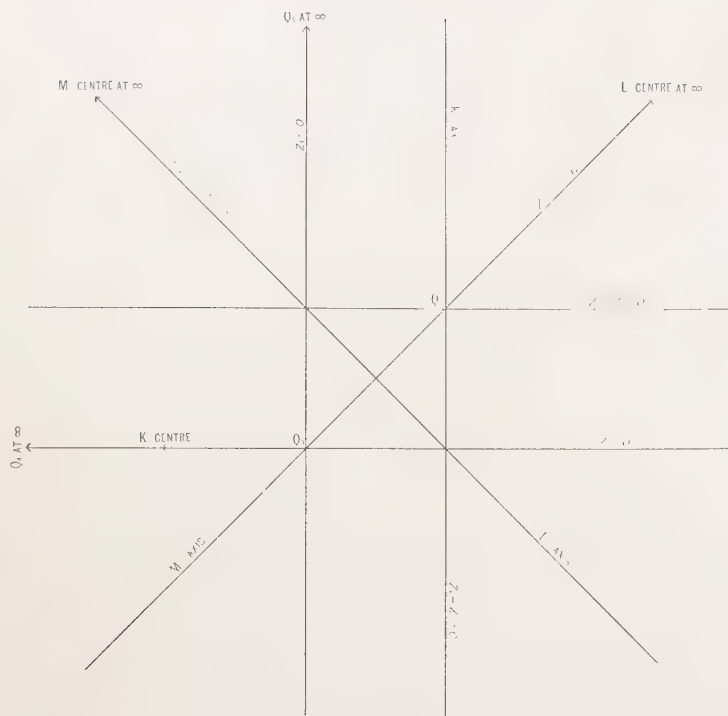


Fig. I.

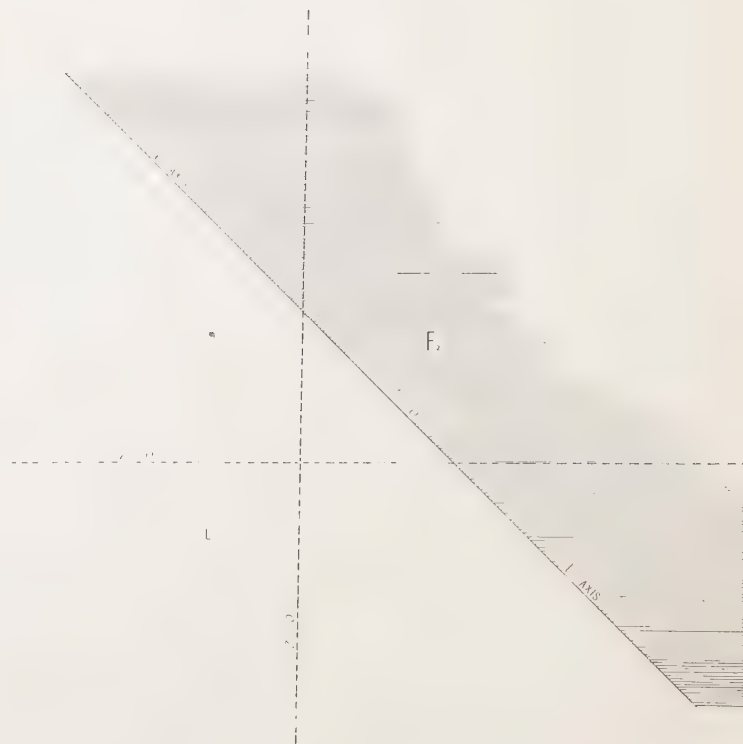


Fig. VII.

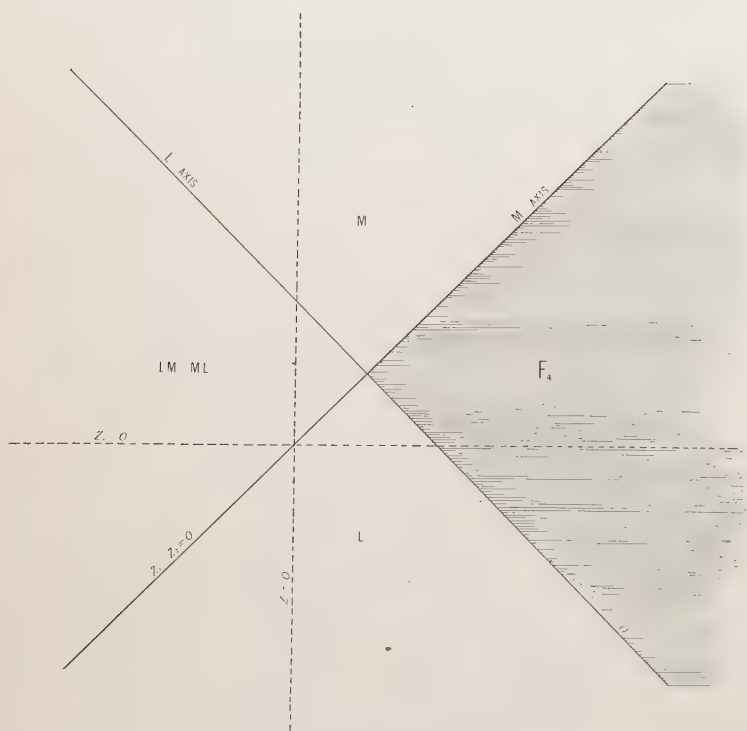


Fig. VIII.

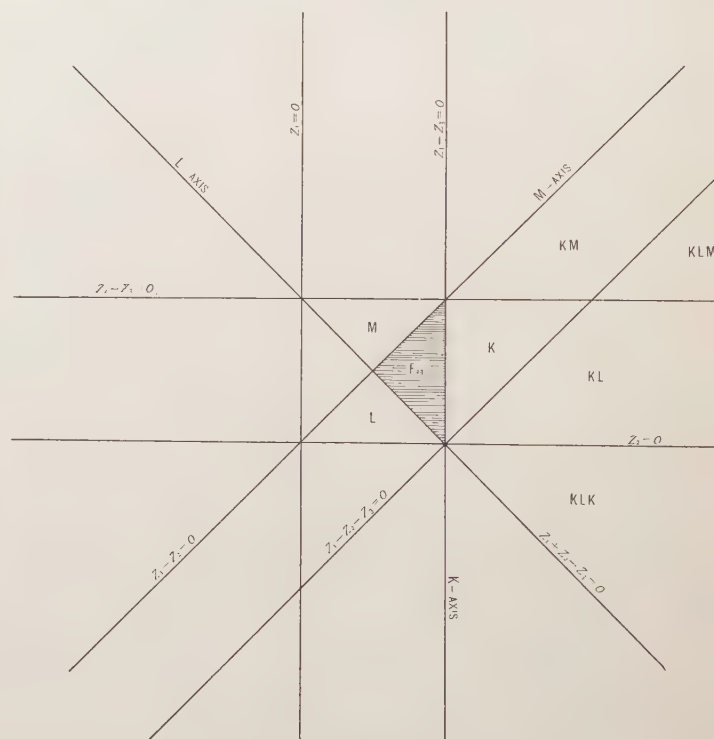


Fig. IX.

The Cross-Ratio Group of 120 Quadratic Cremona Transformations of the Plane.

Part Second:* Complete Form-System of Invariants.

BY HERBERT ELLSWORTH SLAUGHT.

§1.—INVARIANTS OF THE BINARY QUINTIC FORM.

1. Three invariants of the quintic as given by Salmon† are :

$$\left. \begin{aligned} P^2 &= \Pi (12)^2, \\ A &= \Sigma (12)^4 (34)^2 (35)^2 (45)^2, \\ C &= P^2 \Sigma (12)^{-4} (34)^{-2} (35)^{-2} (45)^{-2}, \end{aligned} \right\} \quad (1)$$

where (ij) means the root difference

$$(\alpha_i - \alpha_j), \quad (i, j = 1 \dots 5).$$

2. The ratio $A : P$ written in cross-ratio form is

$$\left. \begin{aligned} A : P &= [3214][3415][3512] + [4123][4325][4521] \\ &+ [5132][5234][5431] + [5213][5314][5412] \\ &- [2314][2415][2513] - [3124][3425][3521] \\ &- [4132][4235][4531] - [5142][5243][5341] \\ &- [5123][5324][5421] - [4213][4315][4512] \end{aligned} \right\} \quad (2)$$

in which

$$[ijkl] = \frac{(\alpha_i - \alpha_k)(\alpha_j - \alpha_l)}{(\alpha_j - \alpha_k)(\alpha_i - \alpha_l)}, \quad (i, j, k, l = 1 \dots 5).$$

If we put in (2)

$$\lambda_1 = [4235],$$

* The first part of this memoir is to be found in the *American Journal of Mathematics*, vol. XXII, pp. 343-388, 1900. It will be referred to here simply as Part First.

† "Modern Higher Algebra," third edition, §241.

and let $\lambda_2 \dots \lambda_5$ be derived from λ_1 by successive application of the cyclic permutation (12345) to the indices, and make use of the relations*

$$\lambda_3 = \frac{1 - \lambda_1}{1 - \lambda_1 \lambda_2}, \quad \lambda_4 = 1 - \lambda_1 \lambda_2, \quad \lambda_5 = \frac{1 - \lambda_2}{1 - \lambda_1 \lambda_2}, \quad (3)$$

we get

$$\begin{aligned} A : P = & - \{ (1 - \lambda_1)^2 [1 + \lambda_2^2 (1 - \lambda_1)^2 + \lambda_1^2 \lambda_2^4] + (1 - \lambda_2)^2 [1 + \lambda_1^2 (1 - \lambda_2)^2 + \lambda_1^4 \lambda_2^2] \\ & + (1 - \lambda_1 \lambda_2)^2 [\lambda_1^2 + \lambda_2^2 + (1 - \lambda_1 \lambda_2)^2 + (1 - \lambda_1)^2 (1 - \lambda_2)^2] \} \\ & : \{ \lambda_1 \lambda_2 (1 - \lambda_1)(1 - \lambda_2)(1 - \lambda_1 \lambda_2) \}. \end{aligned} \quad (4)$$

3. We now identify the roots α_i with the variables† ν_i in the following order :

$$\begin{array}{ccccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \\ \nu_5 & \nu_2 & \nu_3 & \nu_1 & \nu_4 \end{array} \} \quad (5)$$

and obtain the relation

$$\lambda_1 = \rho, \quad \lambda_2 = \frac{\sigma - 1}{\sigma - \rho}. \quad (6)$$

Substituting (6) in (4) and changing to homogeneous coordinates

$$\rho = \frac{z_1}{z_3}, \quad \sigma = \frac{z_2}{z_3},$$

* M. J. M. Hill, "The Anharmonic Ratios of the Roots of a Quintic" (Proceedings of the London Mathematical Society, vol. XIV, p. 182).

† E. H. Moore, "The Cross-Ratio Group of $n!$ Cremona Transformations of Order $n-3$ in Flat Space of $n-3$ Dimensions" (*American Journal of Mathematics*, vol. XXII, p. 280, 1900). For the case $n=5$, I use the variables $\nu_1 \dots \nu_5$, and the fundamental system of cross-ratios,

$$\rho \equiv \rho_4 = [\nu_1 \nu_2 \nu_3 \nu_4] \quad \text{and} \quad \sigma \equiv \rho_5 = [\nu_1 \nu_2 \nu_3 \nu_5].$$

Whence, ρ_1, ρ_2, ρ_3 give the special values $\infty, 0, 1$.

With this notation, for example, the transformation T , corresponding to the substitution on the indices (15)(34), is derived as follows :

$$\begin{aligned} \rho' = \rho_{5243} &= \frac{(\rho_5 - \rho_4)(\rho_2 - \rho_3)}{(\rho_2 - \rho_4)(\rho_5 - \rho_3)} = \frac{\rho - \sigma}{\rho(1 - \sigma)}, \\ \sigma' = \rho_{5241} &= \frac{(\rho_5 - \rho_4)(\rho_2 - \rho_1)}{(\rho_2 - \rho_4)(\rho_5 - \rho_1)} = \frac{\rho - \sigma}{\rho}, \end{aligned}$$

which, in homogeneous coordinates, becomes

$$T; \quad z_1' : z_2' : z_3' = z_3(z_1 - z_2) : (z_1 - z_2)(z_3 - z_2) : z_1(z_3 - z_2).$$

See Part First, Arts. 4, 28 and 58.

there results

$$A: P = - \left\{ \Sigma z_1^2 (z_2 - z_3)^4 + \Sigma z_1^4 (z_2 - z_3)^3 + \Sigma z_1^2 z_2^2 (z_1 - z_2)^2 \right. \\ \left. + (z_1 - z_2)^2 (z_1 - z_3)^2 (z_2 - z_3)^2 \right\} : \{ z_1 z_2 z_3 (z_1 - z_2)(z_1 - z_3)(z_2 - z_3) \}. \quad (7)$$

In a similar manner,

$$C: P^3 = - \left\{ \Sigma z_1^4 z_2^4 (z_1 - z_2)^2 (z_1 - z_3)^4 (z_2 - z_3)^4 + \Sigma z_1^2 z_2^4 z_3^4 (z_1 - z_2)^4 (z_1 - z_3)^4 \right. \\ \left. + \Sigma z_1^2 z_2^2 z_3^4 (z_1 - z_2)^2 (z_1 - z_3)^4 (z_2 - z_3)^4 + z_1^4 z_2^4 z_3^4 (z_1 - z_2)^2 (z_1 - z_3)^2 (z_2 - z_3)^2 \right\} \\ : \{ z_1^3 z_2^3 z_3^3 (z_1 - z_2)^3 (z_1 - z_3)^3 (z_2 - z_3)^3 \}. \quad (8)$$

4. Applying to (7) and (8) the transformation

$$z_1 : z_2 : z_3 = y_1 - y_4 : y_2 - y_4 : y_3 - y_4, \quad (9)$$

we find the functions proportional to A , P^2 , C ;

$$\left. \begin{aligned} \delta_1 A &= \Sigma (y_1 - y_2)^2 (y_3 - y_4)^4 + \Sigma (y_1 - y_2)^2 (y_1 - y_3)^2 (y_2 - y_3)^2, \\ \delta_2 P^2 &= (y_1 - y_2)^2 (y_1 - y_3)^2 (y_1 - y_4)^2 (y_2 - y_3)^2 (y_2 - y_4)^2 (y_3 - y_4)^2, \\ \delta_3 C &= \Sigma (y_1 - y_2)^2 (y_1 - y_3)^4 (y_1 - y_4)^4 (y_2 - y_3)^4 (y_2 - y_4)^4 \\ &\quad + P^2 \Sigma (y_1 - y_4)^2 (y_2 - y_4)^2 (y_3 - y_4)^2. \end{aligned} \right\} \quad (10)$$

5. It is desirable to evaluate A , P^2 and C in terms of the elementary symmetric functions :

$$-p_1 = \Sigma y_i, \quad p_2 = \Sigma y_i y_j, \quad -p_3 = \Sigma y_i y_j y_k, \quad p_4 = y_1 y_2 y_3 y_4, \quad (11) \\ (i, j, k = 1 \dots 4).$$

The results for A and P^2 are easily found :

$$\left. \begin{aligned} \delta_1 A &= 3^2 p_3^2 + 2 \cdot 3 p_2^3 + 2^3 \cdot 5 p_2 p_4 + \dots \text{terms containing } p_1 \text{ as a factor,} \\ \delta_2 P^2 &= 2^8 p_4^3 - 3^3 p_3^4 - 2^2 p_2^3 p_3^2 - 2^7 p_2^2 p_4^2 + 2^4 p_2^4 p_4 + 2^4 \cdot 3^2 p_2 p_3^2 p_4 \\ &\quad + \dots \text{terms containing } p_1 \text{ as a factor.} \end{aligned} \right\} \quad (12)$$

6. For the first part of C , whose weight is 18, we have

$$\left. \begin{aligned} \Sigma (y_1 - y_2)^2 (y_1 - y_3)^4 (y_1 - y_4)^4 (y_2 - y_4)^4 (y_3 - y_4)^4 &\equiv a_1 p_2^9 + a_2 p_3^6 \\ &\quad + a_3 p_3^2 p_4^3 + a_4 p_2^2 p_3^2 + a_5 p_2^3 p_3^4 + a_6 p_2^7 p_4 + a_7 p_2^5 p_4^2 + a_8 p_2^3 p_4^3 \\ &\quad + a_9 p_2 p_4^4 + a_{10} p_2 p_3^4 p_4 + a_{11} p_2^4 p_3^2 p_4 + a_{12} p_2^2 p_3^2 p_4^2 \\ &\quad + 41 \text{ terms containing } p_1 \text{ as a factor.} \end{aligned} \right\} \quad (13)$$

7. In (13) put

$$y_1 = -y_3 = 1, \quad y_2 = y_4 = 0,$$

Whence by (11), $p_1 = p_3 = p_4 = 0, \quad p_2 = -1.$

From which $a_1 = -2^2.$

To determine a_2, a_4, a_5 , put in succession in (13),

$$y_1, y_2, y_3, y_4 = 0, 2, -1, -1; 0, -3, 2, 1; 0, 4, -3, -1.$$

whence

$$p_1, p_2, p_3, p_4 = 0, -3, -2, 0; 0, -7, 6, 0; 0, -13, -12, 0.$$

These substitutions lead to the conditions,

- 1) $2^4 a_2 + 3^6 a_4 - 2^2 \cdot 3^3 a_5 = -2 \cdot 3^8,$
- 2) $2^4 \cdot 3^6 a_2 + 3^2 \cdot 7^6 a_4 - 2^2 \cdot 3^4 \cdot 7^3 a_5 = 2^{11} \cdot 5^5 + 3^4 \cdot 5^5 + 2^{11} \cdot 3^4 \cdot 13 - 7^9,$
- 3) $2^{10} \cdot 3^6 a_2 + 2^2 \cdot 3^2 \cdot \overline{13}^6 a_4 - 2^6 \cdot 3^4 \cdot \overline{13}^3 a_5 = 3^4 5^5 \cdot 7^4 + 2^{11} \cdot 7^4 \cdot 17 + 2^{11} \cdot 3^4 \cdot 5^6 - \overline{13}^9,$

from which

$$a_2 = -2 \cdot 3^5, \quad a_4 = -2 \cdot 3 \cdot 11, \quad a_5 = -2^2 \cdot 3^2 \cdot 11.$$

8. To find a_6, a_7, a_8, a_9 , put in succession,

$$y_1, y_2, y_3, y_4 = 1, -1, 1, -1; 1, -1, 2, -2; 1, -1, 3, -3; 2, -2, 3, -3$$

Whence,

$$p_1, p_2, p_3, p_4 = 0, -2, 0, 1; 0, -5, 0, 4; 0, -10, 0, 9; 0, -13, 0, 36.$$

Then from the equations:

- 1) $2^6 a_6 + 2^4 a_7 + 2^2 a_8 + a_9 = 2^{10},$
- 2) $5^6 a_6 + 2^2 \cdot 5^4 a_7 + 2^4 \cdot 5^2 a_8 + 2^6 \cdot a_9 = -2^6 \cdot 11 \cdot 29 \cdot 113,$
- 3) $2^6 \cdot 5^6 a_6 + 2^4 \cdot 3^2 \cdot 5^4 a_7 + 2^2 \cdot 3^4 \cdot 5^2 a_8 + 3^6 a_9 = -2^{10} \cdot 83 \cdot 687,$
- 4) $\overline{13}^6 a_6 + 2^2 \cdot 3^2 \cdot \overline{13}^4 a_7 + 2^4 \cdot 3^4 \cdot \overline{13}^2 a_8 + 2^6 \cdot 3^6 a_9 = -2^6 \cdot 7 \cdot 238 \cdot 879,$

we get,

$$a_6 = 2^6, \quad a_7 = -2^7 \cdot 11, \quad a_8 = -2^{10} \cdot 7, \quad a_9 = 2^{10} \cdot 47.$$

9. To find $a_3, a_{10}, a_{11}, a_{12}$, put in succession,

$$y_1, y_2, y_3, y_4 = 1, 1, 1, -3; 1, 1, 2, -4; 2, 2, -3, -1; 1, 1, 4, -6,$$

which give,

$$p_1, p_2, p_3, p_4 = 0, -6, 8, -3; 0, -11, 18, -8; 0, -9, 4, 12; 0, -27, 50, -24.$$

Then from the equations:

$$\begin{aligned}
 1) \quad & 3a_3 - 2^7 a_{10} + 2^4 \cdot 3^3 a_{11} - 2^2 \cdot 3^2 a_{12} = 2^7 \cdot 3^3 \cdot 7 \cdot 11, \\
 2) \quad & 2^6 a_3 - 2^2 \cdot 3^4 \cdot 11 a_{10} + \overline{11}^4 a_{11} - 2^3 \cdot \overline{11}^3 a_{12} = 2^3 \cdot 1,339,307, \\
 3) \quad & 2^4 a_3 - 2^4 a_{10} + 3^6 a_{11} + 2^2 \cdot 3^2 a_{12} = 2^3 \cdot 3^2 \cdot 11 \cdot 617, \\
 4) \quad & 2^6 a_3 - 2^2 \cdot 3 \cdot 5^4 a_{10} + 3^{10} a_{11} - 2^3 \cdot 3^5 a_{12} = 2^3 \cdot 3^2 \cdot 7 \cdot 11 \cdot 9,767,
 \end{aligned}$$

we find,

$$a_3 = -2^7 \cdot 3^2 \cdot 23, \quad a_{10} = 2^4 \cdot 3^3, \quad a_{11} = 2^3 \cdot 131, \quad a_{12} = 2^5 \cdot 3^2 \cdot 5.$$

Thus we have found:

$$\left. \begin{aligned}
 \Sigma (y_1 - y_2)^2 (y_1 - y_3)^4 (y_1 - y_4)^4 (y_2 - y_3)^4 (y_2 - y_4)^4 &\equiv -2^2 p_2^9 - 2 \cdot 3^5 p_3^6 \\
 - 2^7 \cdot 3^2 \cdot 23 p_3^2 p_4^3 - 2 \cdot 3 \cdot 11 p_2^6 p_3^2 - 2^2 \cdot 3^2 \cdot 11 p_2^3 p_3^4 + 2^6 p_2^7 p_4 - 2^7 \cdot 11 p_2^5 p_4^2 \\
 - 2^{10} \cdot 7 p_2^3 p_4^3 + 2^{10} \cdot 47 p_2 p_4^4 + 2^4 \cdot 3^3 p_2 p_3^4 p_4 + 2^3 \cdot 131 p_2^4 p_3^2 p_4 + 2^5 \cdot 3^2 \cdot 5 p_2^2 p_3^2 p_4^2 \\
 + \dots \text{ terms containing } p_1 \text{ as a factor.}
 \end{aligned} \right\} (14)$$

10. If the second part of C be evaluated in a similar manner and the result combined with (14), we have finally,

$$\left. \begin{aligned}
 \delta_3 C &= -2^2 p_2^9 - 2 \cdot 3^3 \cdot 23 p_3^6 - 2 \cdot 17 p_2^6 p_3^2 - 2^7 \cdot 151 p_3^2 p_4^3 \\
 &- 2^2 \cdot 73 p_2^3 p_3^4 - 2^6 p_2^7 p_4 + 2^7 p_2^5 p_4^2 - 2^{10} \cdot 13 p_2^3 p_4^3 \\
 &+ 2^{10} \cdot 5 \cdot 11 p_2 p_4^4 + 2^4 \cdot 3^2 \cdot 5^2 p_2 p_3^4 p_4 + 2^3 \cdot 3^3 p_2^4 p_3^2 p_4 + 2^5 \cdot 7 \cdot 11 p_2^2 p_3^2 p_4^2 \\
 &+ \dots \text{ terms containing } p_1 \text{ as a factor.}
 \end{aligned} \right\} (15)$$

§2.—INVARIANTS OF THE SUBGROUP $G_{24}^{(1)}$.

Critical and Non-Critical Points. Arts. 11–13.

11. One of the points

$$1i \quad (i = 2 \dots 5)$$

is *non-critical* in the following cases:

(a). In general, for all the linear transformations of G_{120} which form the subgroup $G_{24}^{(1)}$ and which *permute* these four points among themselves in $4!$ ways.

(b). In particular, for a certain dihedron subgroup of $G_{24}^{(1)}$, which leaves the corresponding *point fixed*. Thus the point $1i$ is fixed under

$$G_6^{1i} \sim \{jkl\} \text{ all,} \quad (i, j, k, l = 2 \dots 5).$$

(c). For such quadratic transformations as leave the *point fixed*, thus $1i$ is fixed under each of the 6 quadratic transformations of the set [Part First, Art. 10],

$$D_{ii}^{-1} \sim \{jkl\} \text{ all } (1i).$$

These belong to the set S_{ii}^{-1} , where

$$S_{ii} \sim \{2345\} \text{ all } (1i) \quad [\text{Part First, Art. 15}]$$

is the complete set of transformations through any one of which $G_{24}^{(1)}$ is transformed into $G_{24}^{(i)}$.

12. Under *all other quadratic transformations* the point $1i$ is *critical*; that is, it must be regarded as a pencil of directions which goes into a range of points on one of the fundamental sides. [Part First, Art. 12.] These transformations consist of

(a) the remaining 18 in the set S_{ii}^{-1} .

(b) all of the sets, S_{ij}^{-1} , ($j \neq i = 2 \dots 5$).

In all, $3 \cdot 24 + 18 = 90$.

13. *A pencil, $1i$, is invariant* under such transformations as permute among themselves its infinity of direction tangents. Evidently this will happen if, and only if, the *corresponding point is fixed*.

As just shown, the point $1i$ is fixed under the *linear* transformations G_6^{1i} and under the *quadratic* transformations of the set D_{ii}^{-1} , and hence the pencil $1i$ is invariant under the linear subgroup,

$$G_6^{1i} \sim \{jkl\} \text{ all,}$$

and under the *quadratic subgroup*,

$$G_{12}^{1i} \sim \{jkl\} \text{ all } \{1i\}.*$$

This is in agreement with Part First, Art. 33, where the 4 pencils and 6 sides were found to form a system of 10 conjugate elements under G_{120} .

The linear subgroup G_6^{1i} , which plays an important rôle in the sequel, is also a subgroup of the quadratic group $G_{24}^{(i)}$, its transformations being the *only linear* transformations in that subgroup.

* See foot-note to Part First, Art. 10.

The Complete Form-System for $G_{24}^{(1)}$. Arts. 14-17.

14. The linear subgroup $G_{24}^{(1)}$ is projectively connected with Klein's collineation group G_{41} by the transformation*

$$\left. \begin{array}{l} \text{direct; } z_1 : z_2 : z_3 = x_2 + x_3 : x_3 + x_1 : x_1 + x_2, \\ \text{inverse; } x_1 : x_2 : x_3 = -z_1 + z_2 + z_3 : z_1 - z_2 + z_3 : z_1 + z_2 - z_3, \end{array} \right\} \quad (1)$$

where $x_1 \dots x_3$ are homogeneous point coordinates; or by the transformation†

$$\left. \begin{array}{l} \text{direct; } z_1 : z_2 : z_3 = y_1 - y_4 : y_2 - y_4 : y_3 - y_4, \\ \text{inverse; } y_1 : y_2 : y_3 : y_4 = 3z_1 - z_2 - z_3 : -z_1 + 3z_2 - z_3 : \\ \quad : -z_1 - z_2 + 3z_3 : -z_1 - z_2 - z_3, \end{array} \right\} \quad (2)$$

where $y_1 \dots y_4$ are supernumerary homogeneous point coordinates.

15. The complete form-system of invariants of Klein's collineation group G_4 consists of the elementary symmetric functions

$$\Sigma y_i y_j, \quad \Sigma y_i y_j y_k, \quad \Pi y_i, \quad (i, j, k = 1 \dots 4) \quad (3)$$

with the identical relation

$$\Sigma y_i = 0.$$

Hence, the complete form-system of invariants of $G_{24}^{(1)}$ will be derived from that of G_{41} by applying the transformation (2) to the forms (3). The results are :

$$\left. \begin{array}{l} p_2 = \Sigma y_i y_j = -6\Sigma z_1^2 + 4\Sigma z_1 z_2 \\ -p_3 = \Sigma y_i y_j y_k = 8\Sigma z_1^3 - 8\Sigma z_1^2 z_2 + 16z_1 z_2 z_3, \\ p_4 = \Pi y_i = -3\Sigma z_1^4 + 4z_1^3 z_2 - 20\Sigma z_1^2 z_2 z_3 + 14\Sigma z_1^2 z_2^2, \end{array} \right\} \quad (4)$$

where again $\Sigma y_i = 0$ identically in the z 's.

*Part First, (5), Art. 7.

† For the case G_{41}^2 , I introduce supernumerary coordinates conveniently in the form

$$y_1 : y_2 : y_3 : y_4 = -x_1 + x_2 + x_3 : x_1 - x_2 + x_3 : x_1 + x_2 - x_3 : -x_1 - x_2 - x_3,$$

from which, in combination with (1), we derive (2). It thus appears that

$$\Sigma y_i = 0$$

identically in the z 's as well as in the x 's. I thus have the y 's related to the z 's of $G_{24}^{(1)}$ just as Professor Moore has related the y 's to the x 's of G_{41} . See his paper, "Concerning Klein's Group of $(n+1)!$ n -ary Collineations" (*American Journal of Mathematics*, vol. XXII, pp. 336-342, 1900).

In these forms put

$$\Sigma z_1 = p, \quad \Sigma z_1 z_2 = q, \quad z_1 z_2 z_3 = r, \quad (5)$$

and they become

$$\left. \begin{aligned} p_2 &= -2 [3p^2 - 2^3 q] \\ -p_3 &= 2^3 [p^3 - 2^2 pq + 2^3 r] \\ p_4 &= -p [3p^3 - 2^4 pq + 2^6 r]. \end{aligned} \right\} \quad (6)$$

16. The forms A , P^2 , C , initially given as functions of the roots of the quintic, have been interpreted (a) as functions of the homogeneous coordinates of the transformations of the cross-ratio group G_{120} , and hence of the subgroup $G_{24}^{(1)}$, by the agreement (5), Art. 3; (b) as functions of the supernumerary homogeneous coordinates of the transformations of G_{41} , by virtue of the substitutions (9), Art. 4, since this is the same as (2), Art. 14; and (c) as rational integral functions of p_1, p_2, p_3, p_4 by (12), Art. 5, and (15), Art. 10.

Hence, by Art. 15, they are *absolute invariants* of $G_{24}^{(1)}$. They may be further simplified by substituting in them the value of p_2, p_3, p_4 from (6), Art. 15, and remembering that $p_1 = 0$, thus

$$\delta_1 A = 2p^2 q^2 - 2 \cdot 3 (p^3 r + q^3) + 19pqr - 3^2 r^2, \quad (7)$$

$$\delta_2 P^2 = p^2 q^2 r^2 - 2^2 (p^3 r^3 + q^3 r^2) + 2 \cdot 3^2 pqr^3 - 3^3 r^4, \quad (8)$$

$$\left. \begin{aligned} \delta_3 C &= 2^2 \cdot 5^2 p^4 q^4 r^2 - 2 \cdot 3^3 \cdot 23r^6 + 2^4 \cdot 5 \cdot 7 p^3 q^3 r^3 - 2 \cdot 5^2 \cdot 43 p^2 q^2 r^4 \\ &\quad + 2 \cdot 3^2 \cdot 157 pqr^5 - 2 \cdot 143 (p^2 q^5 r^2 + p^5 q^2 r^3) - 2 \cdot 17 (p^6 r^4 + q^6 r^2) \\ &\quad - 2^2 \cdot 73 (p^3 r^5 + q^3 r^4) + (p^2 q^8 + p^8 q^2 r^2) + 2 \cdot 5 \cdot 53 (p^4 q r^4 + p q^4 r^3) \\ &\quad + 2 \cdot 5^2 (p^7 q r^3 + p q^7 r) - 2^2 (p^9 r^3 + q^9) - 2^2 \cdot 3 (p^6 q^3 r^2 + p^3 q^6 r). \end{aligned} \right\} \quad (9)$$

17. The form P^2 is of special interest later.

In terms of $y_1 \dots y_4$ for the group G_{41} , we have by (10), Art. 4,

$$P_y = \Pi (y_1 - y_2) \equiv \surd \Delta, \quad (10)$$

and in terms of $z_1 \dots z_3$ for the group $G_{24}^{(1)}$, by (7), Art. 3,

$$P_z = z_1 z_2 z_3 (z_1 - z_2)(z_1 - z_3)(z_2 - z_3), \quad (11)$$

in which latter the factors on the right give the six sides of the quadrangle [Fig. II, Part First].

By well-known principles,* the linear homogeneous substitution group $G_{n!}$, which is isomorphic with the symmetric permutation group on n letters, has one, and only one, *fundamental relative invariant*, namely, $\sqrt[n]{\Delta}$, since (except for $n = 4$) it has one, and only one, self-conjugate subgroup $G_{\frac{n!}{2}}$.

Since the index of this subgroup is 2, it follows that the only relative invariants under $G_{n!}$ ($n \neq 4$) must throw off the factor (-1) .

This conclusion, however, holds also for $n = 4$, since $G_{4!}$ can be generated by transformations *all of period 2*,

$$K', L', M', \quad [\text{Part First, (3), Art. 6.}]$$

so that if any primitive root of unity higher than the second could be thrown off, it would have to be built out of the factors $(+1)$ and (-1) , which is impossible.

Hence, P_y is the only fundamental relative invariant under $G_{4!}$, from which it follows that P_z is the only such form under $G_{24}^{(1)}$.

Therefore, all relative invariant forms under $G_{24}^{(1)}$ must be of the form

$$P^{2\kappa+1} \cdot f(p_2, p_3, p_4), \quad (12)$$

where f is a rational integral function.

§3.—CHARACTERISTICS OF INVARIANTS UNDER G_{120} .

Fundamental Notions and Definitions. Arts. 18–21.

18. Since a quadratic transformation, when applied to any function of the z 's, must double its degree, it follows that no such function can be *invariant in the ordinary sense under G_{120}* .

The only invariant form possible is a *rational fraction* such that, under any quadratic transformation of the group, a common factor (a function of the z 's) is thrown off in numerator and denominator.

Evidently, the degree of the numerator must be the same as that of the denominator and equal to that of the factor thrown off.

19. Since a linear transformation does not change the degree of the function operated upon, it follows that numerator and denominator of an invariant fraction must each be an absolute or relative invariant under $G_{24}^{(1)}$.

* Weber, "Lehrbuch der Algebra," vol. II, p. 161–164.

Such a fraction cannot have both terms relative invariants under $G_{24}^{(1)}$, for [Art. 17, (12)] it would have the form

$$\frac{P^{2\kappa_1+1} \cdot \phi_1(p_2, p_3, p_4)}{P^{2\kappa_2+1} \cdot \phi_2(p_2, p_3, p_4)},$$

which reduces, according as $\kappa_1 \geq \kappa_2$, to

$$(I) \quad \frac{P^{2(\kappa_1-\kappa_2)} \cdot \phi_1(p_2, p_3, p_4)}{\phi_2(p_2, p_3, p_4)} \text{ or } (II) \quad \frac{\phi_1(p_2, p_3, p_4)}{P^{2(\kappa_2-\kappa_1)} \cdot \phi_2(p_2, p_3, p_4)},$$

in which both numerator and denominator are absolute invariants under $G_{24}^{(1)}$. [Art. 16.]

If, then, either numerator or denominator alone is a relative invariant under $G_{24}^{(1)}$, the fraction will have one of the forms

$$(III) \quad \frac{P^{2\kappa_1+1} \cdot \phi_1(p_2, p_3, p_4)}{\phi_2(p_2, p_3, p_4)}, \quad (IV) \quad \frac{\phi_1(p_2, p_3, p_4)}{P^{2\kappa_2+1} \cdot \phi_2(p_2, p_3, p_4)}.$$

Forms of the types (III) and (IV) will be considered at the end of the paper [Art. 52].

In the succeeding investigation, the forms considered for numerator and denominator of invariant fractions will be absolute invariants under $G_{24}^{(1)}$, so that every such fraction will be of the type

$$(V) \quad \frac{\phi_1(p_2, p_3, p_4)}{\phi_2(p_2, p_3, p_4)}.$$

20. It follows at once that an invariant fraction cannot have its numerator and denominator of odd degree in z_1, z_2, z_3 , for it would then have the form

$$\frac{p_3^{2\kappa_1+1} \cdot \phi_1(p_2, p_3^2, p_4)}{p_3^{2\kappa_2+1} \cdot \phi_2(p_2, p_3^2, p_4)},$$

since p_3 is the only fundamental invariant form of *odd* degree in z_1, z_2, z_3 under the linear subgroup $G_{24}^{(1)}$.

But such a fraction, when reduced, becomes, according as $\kappa_1 \geq \kappa_2$,

$$(VI) \quad \frac{p_3^{2(\kappa_1-\kappa_2)} \cdot \phi_1(p_2, p_3^2, p_4)}{\phi_2(p_2, p_3^2, p_4)} \text{ or } (VII) \quad \frac{\phi_1(p_2, p_3^2, p_4)}{p_3^{2(\kappa_2-\kappa_1)} \cdot \phi_2(p_2, p_3^2, p_4)},$$

in which both terms are of *even* degree in z_1, z_2, z_3 .

Thus, numerator and denominator of an invariant fraction must be rational, integral functions of p_2, p_3^2, p_4 , and hence also of p, q, r , and, therefore, symmetric functions of z_1, z_2, z_3 owing to the relations (5) and (6), Art. 15.

21. From these considerations come the following definitions:

(a). An invariant under a quadratic operator of G_{120} is a fraction whose numerator and denominator are rational integral functions of z_1, z_2, z_3 such that it is transformed into itself by the operator, after throwing off a factor in the z 's common to numerator and denominator.

(b). An invariant under the Group G_{120} is a fraction which is invariant under every one of a system of generators of the group, and so under every transformation of the group. For this purpose we use the generators of $G_{24}^{(1)}$,

$$K \sim (34), \quad L \sim (23)(45), \quad M \sim (45),$$

and as the extender to G_{120} , the quadratic inversion,

$$T' \sim (12), \quad [\text{Arts. 4, 28, Part First.}]$$

(c). Any homogeneous function of z_1, z_2, z_3 , which is suitable to form the numerator or denominator of an invariant fraction as above defined, is called an *invariant form*. Evidently such a form may be composed of factors or terms,* each of which is a simpler *invariant form*.

(d). Those forms by means of which all other invariant forms of the group can be rationally and integrally expressed, are called the *system of fundamental forms*, or the *complete form-system* of the group.

THEOREMS ON INVARIANT FORMS. Arts. 22–25.

Theorem I.

22. The most general invariant form under G_{120} is of degree $6n$ in z_1, z_2, z_3 and throws off the factor r^{2n} under the quadratic generator T' .

Proof. We denote by $f_s(z_1, z_2, z_3)$ any homogeneous function of degree s in z_1, z_2, z_3 , which is invariant under G_{120} and investigate the value of s and the nature of the factor thrown off when f_s is operated upon by the quadratic extender,

$$T' \sim (12); \quad z'_1 : z'_2 : z'_3 = z_2 z_3 : z_1 z_3 : z_1 z_2. \quad (1)$$

* In the case of invariant terms, of course all must have the same "throw-off."

Thus we have* $f_s(z_1, z_2, z_3)_{T'} = f_{2s}(z_2 z_3, z_1 z_3, z_1 z_2).$ (2)

Since f_s is an invariant form by hypothesis, some factor of degree s [Art. 18] in z_1, z_2, z_3 , must be thrown off, thus

$$f_{2s}(z_2 z_3, z_1 z_3, z_1 z_2) = f_s(z_1, z_2, z_3) \phi_s(z_1, z_2, z_3). \quad (3)$$

Now apply T' again to (3) and as the left becomes homogeneous of degree $4s$, we have

$$(z_1 z_2 z_3)^s \cdot f_s(z_1, z_2, z_3) = f_s(z_1, z_2, z_3) \phi_s(z_1, z_2, z_3) \phi_s(z_1, z_2, z_3)_{T'}. \quad (4)$$

Hence, dividing by $f_s(z_1, z_2, z_3)$,

$$(z_1 z_2 z_3)^s = r^s = \phi_s(z_1, z_2, z_3) \phi_s(z_1, z_2, z_3)_{T'}. \quad (5)$$

Therefore, $\phi_s(z_1, z_2, z_3) = r^t \quad (t \leq s)$ (6)

and $\phi_s(z_1, z_2, z_3)_{T'} = r^{2t}. \quad (7)$

Substituting (6) and (7) in (5),

$$r^s = r^{3t}, \quad (8)$$

giving

$$s = 3t. \quad (9)$$

Then

$$s \equiv 0 \pmod{3}.$$

But

$$s \equiv 0 \pmod{2}. \quad [\text{Art. 20.}]$$

Hence,

$$s \equiv 0 \pmod{6}. \quad (10)$$

Then from (6),

$$r^t = \phi_s = \phi_{6n} = (z_1 z_2 z_3)^t,$$

from which it follows,

$$t = 2n. \quad (11)$$

Therefore, the general invariant form, R , is of degree $6n$, and throws off the factor r^{2n} under T' .

Theorem II.

23. *The most general invariant form may be decomposed into two factors,*

$$R_{6n} = P^{2\mu} \cdot R_{6(n-2\mu)},$$

where μ equals zero or a positive integer and $R_{6(n-2\mu)}$ contains no factor of P .

*For convenience, the operator is here written as a subscript to the operand.

Proof: The six factors of P are the six sides of the quadrangle Π_1 . [(11), Art. 17.] Since these form a conjugate system under $G_{24}^{(1)}$, and since R_{6n} is invariant in the ordinary sense under all transformations of $G_{24}^{(1)}$, it follows that if R_{6n} contains *any* factor of P , it must contain *every* such factor, and if any of these factors are repeated all must be repeated equally often.

Moreover, since we are considering only such forms as are *absolute* invariants under $G_{24}^{(1)}$, such a factor cannot involve P to an *odd* power. [Art. 19.]

Hence, R_{6n} contains as a factor either no factor of P or else $P^{2\mu}$, where μ is a positive integer.

Theorem III.

$$24. \text{ The curve } R_{6n} = P_{24}^{2\mu} \cdot R_{6(n-2\mu)} = 0, \quad (1)$$

has a multiple point of order $2(n+\mu)$ at each of the critical points $1i$, ($i=2 \dots 5$).

Proof: Consider the two factors of (1).

(a). $P^{2\mu}$ fulfills all the conditions for an *invariant form*, namely:

It is an absolute invariant under the linear subgroup $G_{24}^{(1)}$. [Art. 16.]

It is of requisite degree in z_1, z_2, z_3 ; that is, $6 \cdot 2\mu$ according to theorem I.

It throws off the proper factor under T' , namely, $r^{2 \cdot 2\mu}$, thus

$$P_{T'}^{2\mu} = [z_1^2 z_2^2 z_3^2 (z_1 - z_2)^2 (z_1 - z_3)^2 (z_2 - z_3)^2]^\mu = r^{4\mu} \cdot P^{2\mu}.$$

The curve

$$P = 0$$

is a degenerate sextic, having three branches through each of the points $1i$, since it represents the six sides of the quadrangle Π_1 . [Art. 17.] Hence,

$$P^{2\mu} = 0$$

has at each of these points a 6μ -ple point.

(b). Hence, the remaining factor

$$R_{6(n-2\mu)}$$

must be an *invariant form*, since the product R_{6n} is such by hypothesis. That is,

$$[R_{6(n-2\mu)}]_{T'} = r^{2(n-2\mu)} \cdot R_{6(n-2\mu)}. \quad (2)$$

From this it follows that the curve

$$R_{6(n-2\mu)} = 0 \quad (3)$$

has the multiple points $1i$ each of order $2(n-2\mu)$.

For if the curve (3) contains any of the points $1i$, it must contain all of them, since they form a conjugate system under $G_{24}^{(1)}$, under which subgroup, of course, $R_{6(n-2\mu)}$ is invariant.

25. That it does contain three of these points follows from the properties of the quadratic transformation T' , whose critical points are the three coordinate vertices and whose critical lines form the triangle of reference [Art. 10, Part First], namely:

Under T' a curve through one vertex goes into another curve through the same vertex and throws off as an extra factor the opposite coordinate side; and, conversely, a curve under T' can throw off a coordinate side as a factor only when it contains the opposite coordinate vertex [special case of Art. 11, Part First].

Now (3) reproduces itself under T' and throws off the factor

$$r^{2(n-2\mu)}$$

which contains all three coordinate sides. Hence the curve contains each of the coordinate vertices and, therefore, all four critical points as just shown.

Since a curve passing once through each of the coordinate vertices throws off, under T' , precisely the factor r , in order to throw off $r^{2(n-2\mu)}$, a curve must have $2(n-2\mu)$ branches through each vertex.

Thus three (and hence all four) of the critical points are multiple points of order $2(n-2\mu)$ on the curve (3).

Therefore, the curve (1)

$$P^{2\mu} \cdot R_{6(n-2\mu)} = 0$$

has each of these points as a multiple point of order

$$[6\mu + 2(n-2\mu)] = 2(n+\mu). \quad (4)$$

§4.—COMPLETE DETERMINATION OF R_{6n} FOR $n = 1, 2, 3$.

General Forms of Degree 6, 12, 18. Arts. 26–35.

26. The determination of the most general invariant forms of any given degree $6n$ suitable for numerator and denominator of invariant fractions, involves three steps.

(1). Set up the most general form of the given degree invariant under $G_{24}^{(1)}$, involving arbitrary coefficients.

(2). Apply the transformation T' by which $G_{24}^{(1)}$ extends to G_{120} .

(3). Determine the arbitrary coefficients in such a way that the required factor, r^{2n} , may divide out and leave the original form.

27. For this purpose, all forms will be expressed in terms of p, q, r , since the application of T' to these is peculiarly simple, namely,

$$\left. \begin{aligned} p_{T'} &= q, \\ q_{T'} &= pr, \\ r_{T'} &= r^2. \end{aligned} \right\} \quad (1)$$

28. The most general invariant form of degree 6 under $G_{24}^{(1)}$ is

$$a_1 p_2^3 + a_2 p_3^2 + a_3 p_2 p_4,$$

which may be expressed in terms of p, q, r in the following manner:*

	(1)	(2)	(3)	(4)	(5)	(6)	(7)
	p^6	$p^4 q$	q^3	$p^3 r$	pqr	r^2	$p^2 q^2$
$- 2^3 a_1$	3^3	$- 2^3 \cdot 3^3$	$- 2^9$	0	0	0	$2^6 \cdot 3^2$
$+ 2^6 a_2$	1	$- 2^3$	0	2^4	$- 2^6$	2^6	2^4
$+ 2a_3$	3^2	$- 2^3 \cdot 3^2$	0	$2^6 \cdot 3$	$- 2^9$	0	2^7
Apply T'	q^6	$pq^4 r$	$p^3 r^3$	$q^3 r^2$	pqr^3	r^4	$p^2 q^2 r^2$
Divide by r^2	..		$p^3 r$	q^3	pqr	r^2	$p^2 q^2$

* In each case the form is written in a rectangular array with the original letters at the top, those resulting from the application of T' next to the bottom, those left after dividing all terms possible by r^{2n} in the bottom row, while the arbitrary coefficients occupy the column at the left and the numerical coefficients are written in the body of the array.

The conditions to be met are—

- (1). The new form must throw off r^2 .
- (2). The resulting quotient must be the same as the original form.

In order to meet these conditions,

- (a). The coefficients of p^6 and p^4q must vanish.
- (b). The coefficients of like terms in the original and resulting forms may be equated [r^2 having been divided out].

These give only *two independent relations*,

$$\begin{aligned} 2^2 \cdot 3^3 a_1 - 2^5 a_2 - 3^2 a_3 &= 0, \\ 2^5 a_1 - 2^3 a_2 - 3 a_3 &= 0. \end{aligned}$$

From which

$$a_1 : a_2 : a_3 = 6 : 9 : 40.$$

Hence the most general invariant of the 6th degree is proportional to

$$2 \cdot 3p_2^3 + 3^2p_3^2 + 2^3 \cdot 5p_2p_4. \quad (2)$$

This is precisely the form A derived from the quintic. [(12), Art. 5.]

Its value in terms of p, q, r may be read from the table

$$\delta A = 2p^2q^2 - 2 \cdot 3(p^3r + q^3) + 19pqr - 3^2r^2, \quad (3)$$

thus agreeing with (7), Art. 16.

29. The most general invariant form of degree 12 under $G_{24}^{(1)}$ is

$$b_1p_2^6 + b_2p_3^4 + b_3p_4^3 + b_4p_2^4p_4 + b_5p_3^2 + b_6p_2^2p_4^2 + b_7p_2p_3^2p_4, \quad (4)$$

which, in terms of p, q, r in rectangular array, is

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(9)	(9)
	p^{12}	p^8q^2	$p^{10}q$	p^6q_3	p^9r	p^7qr	q^3r^2	p^3r^3	p^2q^5
$+ 2^6b_1$	3^6	$2^6 \cdot 3^5 \cdot 5$	$- 2^4 \cdot 3^6$	$- 2^{11} \cdot 3^3 \cdot 5$	0	0	0	0	$- 2^{16} \cdot 3^2$
$+ 2^{12}b_2$	1	$2^5 \cdot 3$	$- 2^4$	$- 2^8$	2^5	$- 2^7 \cdot 3$	0	2^{11}	0
$- b_3$	3^3	$2^8 \cdot 3^2$	$- 2^4 \cdot 3^3$	$- 2^{12}$	$2^6 \cdot 3^3$	$- 2^{11} \cdot 3^2$	0	0	0
$- 2^4b_4$	3^5	$2^7 \cdot 3^3 \cdot 7$	$- 2^4 \cdot 3^5$	$- 2^{13} \cdot 3^2$	$2^6 \cdot 3^4$	$- 2^{11} \cdot 3^3$	0	0	$- 2^{16}$
$- 2^9b_5$	3^3	$2^4 \cdot 3^2 \cdot 19$	$- 2^4 \cdot 3^3$	$- 2^7 \cdot 67 \cdot$	$2^4 \cdot 3^3$	$- 2^6 \cdot 3^4$	$- 2^{15}$	0	$- 2^{13}$
$+ 2^2b_6$	3^4	$2^6 \cdot 3^2 \cdot 13$	$- 2^4 \cdot 3^4$	$- 2^{11} \cdot 3^2$	$2^7 \cdot 3^3$	$- 2^{12} \cdot 3^2$	0	0	0
$+ 2^7b_7$	3^2	$2^4 \cdot 53$	$- 2^4 \cdot 3^2$	$- 2^7 \cdot 17$	$2^4 \cdot 3 \cdot 7$	$- 2^6 \cdot 59$	0	$2^{12} \cdot 3$	0
Apply T'	q^{12}	$p^2q^3r^2$	$pq^{10}r$	$p^3q^6r^3$	q^2r^2	pq^7r^3	p^3r^7	q^3r^6	$p^5q^2r^5$
Divide by r^4							p^3r^3	q^3r^2	p^5q^2r

(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)
p^5q^r	p^4qr^2	pq^4r	q^6	p^6r^2	pqr^3	p^4q^4	$p^2q^2r^2$	p^3q^3r	r^4
0	0	0	2^{18}	0	0	$2^{12} \cdot 3^3 \cdot 5$	0	0	0
$2^9 \cdot 3$	$- 2^{10} \cdot 3$	0	0	$2^7 \cdot 3$	$- 2^{13}$	2^8	$2^{11} \cdot 3$	$- 2^{11}$	2^{12}
$2^{14} \cdot 3$	$- 2^{16} \cdot 3$	0	0	$2^{12} \cdot 3^2$	0	0	0	0	0
$2^{13} \cdot 3^3$	0	2^{18}	0	0	0	$2^{12} \cdot 3^3$	0	$- 2^{17} \cdot 3$	0
$2^9 \cdot 3^2 \cdot 5$	$- 2^9 \cdot 3^3$	2^{15}	0	$2^6 \cdot 3^3$	0	$2^{10} \cdot 13$	$2^{12} \cdot 3^2$	$- 2 \cdot 11$	0
$2^{13} \cdot 3 \cdot 5$	$- 2^{16} \cdot 3$	0	0	$2^{12} \cdot 3^2$	0	2^{14}	2^{18}	$- 2^{17}$	0
$2^9 \cdot 3^3$	$- 2^9 \cdot 7^2$	0	0	$2^6 \cdot 3 \cdot 19$	$- 2^{15}$	2^{11}	$2^{13} \cdot 5$	$- 2^{14}$	0
$p^2q^5r^4$	pq^4r^5	p^4qr^6	p^6r^6	q^6r^4	pqr^7	$p^4q^4r^4$	$p^2q^2r^6$	$p^3q^3r^5$	r^8
p^2q^5	pq^4r	p^4qr^2	p^6r^2	q^6	pqr^3	p^4q^4	$p^2q^2r^2$	p^3q^3r	r^4

30. The conditions to be met here are—

(a). The coefficients of terms (1) to (6) must vanish.

(b). The coefficients of term (7), (8); (9), (10); (11), (12); (13), (14), must be equal in pairs.

(c). The coefficients of terms (15 to (19) are the same in the old and new forms, and hence give no relations.

There are then 10 relations, which are not all independent, but readily reduce to the following 7 equations:

b_1	b_2	b_3	b_4	b_5	b_6	b_7	
$2^6 \cdot 3^6$	$+ 2^{12}$	$- 3^3$	$- 2^4 \cdot 3^5$	$- 2^9 \cdot 3^3$	$+ 2^2 \cdot 3^4$	$+ 2^7 \cdot 3^2$	$= 0$
$2^4 \cdot 3^5$	$+ 2^9 \cdot 3$	$- 3^2$	$- 2^3 \cdot 3^3 \cdot 7$	$- 2^5 \cdot 3^2 \cdot 19$	$+ 3^2 \cdot 13$	$+ 2^3 \cdot 5 \cdot 3$	$= 0$
$2^5 \cdot 3^5 \cdot 5$	$+ 2^8$	$- 1$	$- 2^5 \cdot 3^2$	$- 2^4 \cdot 6 \cdot 7$	$+ 2 \cdot 3^2$	$+ 2^2 \cdot 17$	$= 0$
2^{12}	$- 2^7 \cdot 3$	$+ 3^2$	0	$+ 2^3 \cdot 3^3$	$- 2^2 \cdot 3^2$	$- 2 \cdot 3 \cdot 19$	$= 0$
0	$+ 2^5$	$- 1$	0	$- 2^6$	0	$+ 2 \cdot 3$	$= 0$
$2^8 \cdot 3^2$	$+ 2^7 \cdot 3$	$- 3$	$- 2^3 \cdot 5 \cdot 7$	$- 2^4 \cdot 6 \cdot 1$	$+ 2 \cdot 3 \cdot 5$	$+ 2^2 \cdot 3^3$	$= 0$
0	$- 2^6 \cdot 3$	$+ 3$	$+ 2^6$	$+ 2^2 \cdot 7 \cdot 13$	$- 2^2 \cdot 3$	$- 7^2$	$= 0$

(5)

31. Two invariants of the 12th degree are already known, P^2 [Art. 24], and A^2 [Art. 28].

Hence their values,

p_2^6	p_3^4	p_4^3	$p_2^4 p_4$	$p_2^2 p_3^2$	$p_2^3 p_2^4$	$p_2 p_3^2 p_4$
$A^2 = 2^2 \cdot 3^2$	$+ 3^4$	0	$+ 2^5 \cdot 3 \cdot 5$	$+ 2^2 \cdot 3^3$	$+ 2^6 \cdot 5^2$	$+ 2^4 \cdot 3^2 \cdot 5$
$P^2 = 0$	$- 3^3$	$+ 2^8$	$+ 2^4$	$- 2^2$	$- 2^7$	$+ 2^4 \cdot 3^2$
b_1	b_2	b_3	b_4	b_5	b_6	b_7

(6)

must each satisfy the above system of equations. This is easily verified. Therefore, there exists an infinity of solutions of the system (5) of the form

$$m_1 A^2 + m_2 P^2, \quad (7)$$

where m_1 and m_2 are arbitrary parameters. Hence, all the first minors in the determinant of (5) must vanish, as well as the determinant itself. If, now, any second minor does not vanish, then no solution exists other than those included in the form (7). The second minor obtained by omitting the 5th and 7th columns and the 1st and 2d rows is easily shown to be different from zero.

Hence, all invariant forms of the 12th degree are included in the form (7).

32. The most general form of degree 18 invariant under $G_{24}^{(1)}$ involves 12 arbitrary constants, thus

$$c_1 p_2^9 + c_2 p_3^6 + c_3 p_2^6 p_3^2 + c_4 p_3^2 p_4^3 + c_5 p_2^3 p_3^4 + c_6 p_2^7 p_4 + c_7 p_2^5 p_4^2 + c_8 p_2^3 p_4^3 \\ + c_9 p_2 p_4^4 + c_{10} p_2 p_3^4 p_4 + c_{11} p_2^4 p_3^2 p_4 + c_{12} p_2^2 p_3^2 p_4^2. \quad (8)$$

When expressed in terms of p, q, r in the rectangular array, this becomes

	(1)	(2)	(3)	(4)	(5)	(6)
	p^{18}	$p^{16}q$	$p^{15}r$	$p^{14}q^2$	$p^{12}q^3$	$p^{12}r^2$
$- 2^9c_1$	3^9	$-2^3 \cdot 3^{10}$	0	$2^8 \cdot 3^9$	$-2^{11} \cdot 3^7 \cdot 7$	0
$+ 2^{18}c_2$	1	$-2^3 \cdot 3$	$2^4 \cdot 3$	$2^4 \cdot 3 \cdot 5$	$-2^8 \cdot 5$	$2^6 \cdot 3 \cdot 5$
$+ 2^{12}c_3$	3^6	$-2^3 \cdot 3^7$	$2^4 \cdot 3^6$	$2^4 \cdot 3^5 \cdot 47$	$-2^9 \cdot 3^5 \cdot 5$	$2^6 \cdot 3^6$
$- 2^6c_4$	3^3	$-2^3 \cdot 3^4$	$2^4 \cdot 3^3 \cdot 5$	$2^4 \cdot 3^2 \cdot 43$	$-2^8 \cdot 5 \cdot 23$	$2^6 \cdot 3^2 \cdot 5 \cdot 23$
$- 2^{15}c_5$	3^3	$-2^3 \cdot 3^4$	$2^5 \cdot 3^3$	$2^5 \cdot 3^2 \cdot 23$	$-2^9 \cdot 73$	$2^7 \cdot 3^4$
$+ 2^7c_6$	3^8	$-2^3 \cdot 3^9$	$2^6 \cdot 3^7$	$2^6 \cdot 3^6 \cdot 5 \cdot 7$	$-2^9 \cdot 3^5 \cdot 7 \cdot 11$	0
$- 2^5c_7$	3^7	$-2^3 \cdot 3^8$	$2^7 \cdot 3^6$	$2^7 \cdot 3^5 \cdot 17$	$-2^{10} \cdot 3^4 \cdot 5 \cdot 7$	$2^{12} \cdot 3^5$
$+ 2^3c_8$	3^6	$-2^3 \cdot 3^7$	$2^6 \cdot 3^6$	$2^6 \cdot 3^5 \cdot 11$	$-2^9 \cdot 3^5 \cdot 7$	$2^{12} \cdot 3^5$
$- 2c_9$	3^5	$-2^3 \cdot 3^6$	$2^8 \cdot 3^4$	$2^{11} \cdot 3^3$	$-2^{12} \cdot 3^2 \cdot 7$	$2^{18} \cdot 3^4$
$- 2^{13}c_{10}$	3^2	$-2^3 \cdot 3^3$	$2^5 \cdot 3 \cdot 5$	$2^5 \cdot 67$	$-2^{10} \cdot 11$	$2^7 \cdot 3 \cdot 5^2$
$- 2^{10}c_{11}$	3^5	$-2^3 \cdot 3^6$	$2^4 \cdot 3^4 \cdot 7$	$2^4 \cdot 3^3 \cdot 137$	$-2^8 \cdot 3^3 \cdot 143$	$2^6 \cdot 3^{14} \cdot 19$
$+ 2^8c_{12}$	3^4	$-2^3 \cdot 3^5$	$2^4 \cdot 3^3 \cdot 11$	$2^4 \cdot 3^2 \cdot 7 \cdot 19$	$-2^8 \cdot 3^3 \cdot 43$	$2^6 \cdot 3^3 \cdot 13^3$
Apply T'	q^{18}	$pq^{16}r$	$q^{15}r^2$	$p^2q^{14}r^2$	$p^3q^{12}r^3$	$q^{12}r^4$
Divide by r^6						

	(7)	(8)	(9)	(10)	(11)
	$p^{10}q^4$	p^8q^5	$p^{13}qr$	$p^{11}q^2r$	$p^{10}qr^2$
$- 2^9c_1$	$2^{13} \cdot 3^7 \cdot 7$	$- 2^{16} \cdot 3^6 \cdot 7$	0	0	0
$+ 2^{12}c_2$	$2^8 \cdot 3 \cdot 5$	$- 2^{11} \cdot 3$	$- 2^6 \cdot 3 \cdot 5$	$2^9 \cdot 3 \cdot 5$	$- 2^{10} \cdot 3^5$
$+ 2^{12}c_3$	$2^{10} \cdot 3^3 \cdot 5 \cdot 29$	$- 2^{20} \cdot 3^2$	$- 2^6 \cdot 3^6 \cdot 5$	$2^{13} \cdot 3^5$	$- 2^{10} \cdot 3^6$
$- 2^6c_4$	$2^{12} \cdot 17$	$- 2^{16}$	$- 2^6 \cdot 3^2 \cdot 71$	$2^{11} \cdot 3 \cdot 47$	$- 2^{10} \cdot 3 \cdot 7 \cdot 43$
$- 2^{15}c_5$	$2^8 \cdot 491$	$- 2^{11} \cdot 3 \cdot 41$	$- 2^7 \cdot 3^3 \cdot 5$	$2^9 \cdot 3^2 \cdot 31$	$- 2^{11} \cdot 3^4$
$+ 2^7c_6$	$2^{12} \cdot 3^5 \cdot 5 \cdot 7$	$- 2^{15} \cdot 3^3 \cdot 7 \cdot 13$	$- 2^9 \cdot 3^6 \cdot 7$	$2^{12} \cdot 3^6 \cdot 7$	0
$- 2^5c_7$	$2^{12} \cdot 3^3 \cdot 5 \cdot 17$	$- 2^{15} \cdot 3^2 \cdot 61$	$- 2^{10} \cdot 3^5 \cdot 7$	$2^{15} \cdot 3^4 \cdot 5$	$- 2^{15} \cdot 3^4 \cdot 5$
$+ 2^3c_8$	$2^{13} \cdot 3^3 \cdot 11$	$- 2^{17} \cdot 3^3$	$- 2^9 \cdot 3^5 \cdot 7$	$2^{12} \cdot 3^4 \cdot 19$	$- 2^{15} \cdot 3^4 \cdot 5$
$- 2c_9$	$2^{16} \cdot 3^2$	$- 2^{19}$	$- 2^{11} \cdot 3^3 \cdot 7$	$2^{15} \cdot 3^4$	$- 2^{16} \cdot 3^3 \cdot 5$
$- 2^{13}c_{10}$	$2^3 \cdot 3 \cdot 43$	$- 2^{11} \cdot 5^2$	$- 2^7 \cdot 73$	$2^9 \cdot 3 \cdot 47$	$- 2^{11} \cdot 71$
$- 2^{10}c_{11}$	$2^{11} \cdot 3^2 \cdot 59$	$- 2^{15} \cdot 5 \cdot 13$	$- 2^6 \cdot 3^3 \cdot 7 \cdot 11$	$2^{11} \cdot 3^3 \cdot 5^2$	$- 2^{10} \cdot 3^3 \cdot 53$
$+ 2^8c_{12}$	$2^{10} \cdot 277$	$- 2^{15} \cdot 13$	$- 2^6 \cdot 3^2 \cdot 157$	$2^{12} \cdot 3 \cdot 5 \cdot 11$	$- 2^{10} \cdot 3 \cdot 11 \cdot 41$
Apply T'	$p^4q^{10}r^4$	$p^5q^8r^5$	$pq^{13}r^3$	$p^2q^{11}r^4$	$pq^{10}r^5$
Divide by r^6					

	(12)	(13)	(14)	(15)	(16)	(17)	(18)
	$p^9 q^3 r$	q^9	$p^9 r^3$	$p^6 r^4$	$q^6 r^2$	$p^4 q^7$	$p^7 q^4 r$
$- 2^9 c_1$	0	$- 2^{27}$	0	0	0	$- 2^{23} \cdot 3^4$	0
$+ 2^{18} c_2$	$- 2^{11} \cdot 3 \cdot 5$	0	$2^{11} \cdot 5$	$2^{12} \cdot 3 \cdot 5$	0	0	$2^{12} \cdot 3 \cdot 5$
$+ 2^{12} c_3$	$- 2^{12} \cdot 3 \cdot 5 \cdot 17$	0	0	0	2^{24}	$- 2^{20} \cdot 11 \cdot$	$2^{16} \cdot 3^4 \cdot 5$
$- 2^6 c_4$	$- 2^{14} \cdot 5 \cdot 11$	0	$2^{12} \cdot 5 \cdot 47$	$2^{18} \cdot 5^2$	0	0	2^{20}
$- 2^{15} c_5$	$- 2^{11} \cdot 5 \cdot 61$	0	$2^{11} \cdot 3^3$	$2^{12} \cdot 3^3$	0	$- 2^{17}$	$2^{14} \cdot 3 \cdot 31$
$+ 2^7 c_6$	$- 2^{15} \cdot 3^4 \cdot 5 \cdot 7$	0	0	0	0	$- 2^{21} \cdot 3^2 \cdot 5$	$2^{18} \cdot 3 \cdot 5 \cdot 7$
$- 2^5 c_7$	$- 2^{17} \cdot 3^4 \cdot 5$	0	0	0	0	$- 2^{23}$	$2^{19} \cdot 3^2 \cdot 5$
$+ 2^3 c_8$	$- 2^{15} \cdot 3^3 \cdot 5^2$	0	$2^{18} \cdot 3^3$	0	0	0	$2^{22} \cdot 3^2$
$- 2 c_9$	$- 2^{19} \cdot 3 \cdot 5$	0	$2^{20} \cdot 3^2$	$2^{24} \cdot 3$	0	0	2^{23}
$- 2^{13} c_{10}$	$- 2^{11} \cdot 3^3 \cdot 5$	0	$2^{11} \cdot 3^2 \cdot 5$	$2^{12} \cdot 3 \cdot 5 \cdot 7$	0	0	2^{19}
$- 2^{10} c_{11}$	$- 2^{13} \cdot 3 \cdot 5 \cdot 47$	0	$2^{12} \cdot 3^4$	0	0	$- 2^{20}$	$2^{16} \cdot 5 \cdot 41$
$+ 2^8 c_{12}$	$- 2^{12} \cdot 5 \cdot 11^2$	0	$2^{13} \cdot 3^2 \cdot 11$	$2^{18} \cdot 3^2$	0	0	$2^{18} \cdot 17$
Apply T'	$p^3 q^9 r^5$	$p^9 r^9$	$q^9 r^6$	$q^6 r^8$	$p^6 r^{10}$	$p^7 q^4 r^7$	$p^4 q^7 r^6$
Divide by r^6		$p^9 r^3$	q^9	$q^6 r^2$	$p^6 r^4$	$p^7 q^4 r$	$p^4 q^7$

	(19)	(20)	(21)	(22)	(23)	(24)
	p^3r^5	q^3r^4	p^2q^8	$p^8q^2r^2$	p^7qr^3	pq^7r
$- 2^9c_1$	0	0	$2^{24} \cdot 3^3$	0	0	0
$+ 2^{18}c_2$	$2^{16} \cdot 3$	0	0	$2^{11} \cdot 3^2 \cdot 5$	$- 2^{12} \cdot 3 \cdot 5$	0
$+ 2^{12}c_3$	0	0	2^{22}	$2^{12} \cdot 3^5 \cdot 5$	0	$- 2^{24}$
$- 2^6c_4$	2^{24}	0	0	$2^{14} \cdot 3^2 \cdot 29$	$- 2^{17} \cdot 67$	0
$- 2^{15}c_5$	0	$- 2^{21}$	0	$2^{11} \cdot 3^3 \cdot 19$	$- 2^{13} \cdot 3^4$	0
$+ 2^7c_6$	0	0	2^{26}	0	0	$- 2^{27}$
$- 2^5c_7$	0	0	0	$2^{19} \cdot 3^3 \cdot 5$	0	0
$+ 2^3c_8$	0	0	0	$2^{18} \cdot 3^5$	$- 2^{21} \cdot 3^3$	0
$- 2c_9$	0	0	0	$2^{22} \cdot 3^2$	$- 2^{23} \cdot 3^2$	0
$- 2^{13}c_{10}$	$2^{18} \cdot 3$	0	0	$2^{11} \cdot 3 \cdot 7 \cdot 19$	$- 2^{13} \cdot 3 \cdot 41$	0
$- 2^{10}c_{11}$	0	0	0	$2^{13} \cdot 3^4 \cdot 13$	$- 2^{17} \cdot 3^3$	0
$+ 2^8c_{12}$	0	0	0	$2^{12} \cdot 5 \cdot 353$	$- 2^{19} \cdot 3 \cdot 5$	0
Apply T'	q^3r^{10}	p^3r^7	$p^8q^2r^8$	$p^2q^8r^6$	pq^7r^7	p^7qr^9
Divide by r^6	q^3r^4	p^3r	$p^8q^2r^2$	p^2q^8	pq^7r	p^7qr^3

	(25)	(26)	(27)	(28)	(29)	(30)	(31)
	$p^6q^3r^2$	p^3q^6r	$p^5q^2r^3$	$p^2q^5r^2$	p^4qr^4	pq^4r^3	r^6
-2^9c_1	0	0	0	0	0	0	0
$+2^{18}c_2$	$-2^{14} \cdot 3 \cdot 5$	0	$2^{15} \cdot 3 \cdot 5$	0	$-2^{15} \cdot 3 \cdot 5$	0	2^{18}
$+2^{12}c_3$	$-2^{17} \cdot 3^3 \cdot 5$	$2^{23} \cdot 5$	0	$-2^{22} \cdot 3^2$	0	0	0
-2^6c_4	$-2^{18} \cdot 5^2$	0	$2^{20} \cdot 19$	0	$-2^{22} \cdot 7$	0	0
$-2^{15}c_5$	$-2^{14} \cdot 3 \cdot 67$	2^{20}	$2^{16} \cdot 3^2 \cdot 5$	$-2^{20} \cdot 3$	$-2^{15} \cdot 3^3$	2^{22}	0
$+2^7c_6$	0	$2^{24} \cdot 3 \cdot 7$	0	0	0	0	0
-2^5c_7	$-2^{22} \cdot 3^2 \cdot 5$	2^{26}	0	-2^{27}	0	0	0
$+2^3c_8$	$-2^{21} \cdot 3^2 \cdot 7$	0	$2^{24} \cdot 3^2$	0	0	0	0
$-2c_9$	$-2^{24} \cdot 3$	0	2^{27}	0	-2^{27}	0	0
$-2^{13}c_{10}$	$-2^{14} \cdot 3 \cdot 41$	0	$2^{16} \cdot 5 \cdot 11$	0	$-2^{15} \cdot 89$	0	0
$-2^{10}c_{11}$	$-2^{23} \cdot 3$	2^{23}	$2^{11} \cdot 3^3$	$-2^{22} \cdot 5$	0	2^{24}	0
$+2^{18}c_{12}$	$-2^{17} \cdot 5^3$	0	$2^{19} \cdot 47$	0	$-2^{22} \cdot 3$	0	0
Apply T'	$p^3q^6r^7$	$p^6q^3r^8$	$p^2q^5r^8$	$p^5q^2r^9$	pq^4r^9	$p^4q^4r^{10}$	r^{12}
Divide by r^6	p^3q^6r	$p^6q^3r^2$	$p^2q^5r^2$	$p^5q^2r^3$	pq^4r^3	$p^4q^4r^4$	r^6

	(32)	(33)	(34)	(35)	(36)	(37)
	p^6q^6	p^5q^5r	$p^4q^4r^2$	$p^3q^3r^3$	$p^2q^2r^4$	pqr^5
$- 2^0c_1$	$2^{20} \cdot 3^4 \cdot 7$	0	0	0	0	0
$+ 2^{18}c_2$	2^{12}	$- 2^{14} \cdot 3$	$2^{14} \cdot 3 \cdot 5$	$- 2^{17} \cdot 5$	$2^{16} \cdot 3 \cdot 5$	$- 2^{18} \cdot 3$
$+ 2^{12}c_3$	$2^{16} \cdot 211$	$- 2^{18} \cdot 3^2 \cdot 19$	$2^{18} \cdot 3 \cdot 5$	0	0	0
$- 2^6c_4$	0	0	0	0	0	0
$- 2^{15}c_5$	$2^{14} \cdot 17$	$- 2^{17} \cdot 3 \cdot 5$	$2^{17} \cdot 3 \cdot 13$	$- 2^{19} \cdot 11$	$2^{18} \cdot 3^2$	0
$+ 2^7c_6$	$2^{18} \cdot 3^2 \cdot 7^2$	$- 2^{21} \cdot 3^3 \cdot 7$	0	0	0	0
$- 2^5c_7$	$2^{21} \cdot 3^2$	$- 2^{22} \cdot 3 \cdot 11$	$2^{24} \cdot 3 \cdot 5$	0	0	0
$+ 2^3c_8$	2^{21}	$- 2^{23} \cdot 3$	$2^{25} \cdot 3$	$- 2^{27}$	0	0
$- 2c_9$	0	0	0	0	0	0
$- 2^{13}c_{10}$	2^{15}	$- 2^{17} \cdot 3$	$2^{18} \cdot 7$	$- 2^{22}$	$2^{19} \cdot 3^2$	$- 2^{21}$
$- 2^{10}c_{11}$	$2^{16} \cdot 5 \cdot 7$	$- 2^{18} \cdot 3^2 \cdot 7$	$2^{18} \cdot 139$	$- 2^{23} \cdot 3$	0	0
$+ 2^8c_{12}$	2^{18}	$- 2^{20} \cdot 3$	$2^{20} \cdot 13$	$- 2^{23} \cdot 3$	2^{24}	0
Apply T'	$p^6q^6r^6$	$p^5q^5r^7$	$p^4q^4r^8$	$p^3q^3r^9$	$p^2q^2r^{10}$	pqr^{11}
Divide by r^6	p^6q^6	p^5q^5r	$p^4q^4r^2$	$p^3q^3r^3$	$p^2q^2r^4$	pqr^5

33. The conditions in this case are—

(a). The coefficients of terms (1) to (12) must vanish.

(b). The coefficients of terms (13) to (30) must be equal in pairs; that is,

$$(13), (14); (15), (16); \dots (29), (30).$$

(c). The coefficients of terms (31) to (37) are the same in the new form as in the original, and hence afford no relations.

These conditions give rise to 21 relations among the 12 c 's. This system reduces readily to the following 16 equations:

	c_1	c_2	c_3	c_4	c_5	c_6
(1)	$2^8 \cdot 3^9$	-2^{17}	$-2^{11} \cdot 3^6$	$+2^5 \cdot 3^3$	$+2^{14} \cdot 3^3$	$-2^6 \cdot 3^8$
(2)	0	$+2^{13}$	$+2^7 \cdot 3^5$	$-2 \cdot 3^2 \cdot 5$	$-2^{11} \cdot 3^2$	$+2^4 \cdot 3^6$
(3)	$2^8 \cdot 3^7 \cdot 7$	$-2^{14} \cdot 5$	$-2^8 \cdot 3^3 \cdot 157$	$+2^2 \cdot 5 \cdot 23$	$+2^{12} \cdot 73$	$-2^4 \cdot 3^5 \cdot 7 \cdot 11$
(4)	0	$+2^{12} \cdot 5$	$+2^6 \cdot 3^5$	$-3 \cdot 5 \cdot 23$	$-2^{10} \cdot 3^3$	0
(5)	$2^7 \cdot 3^6 \cdot 7$	$-2^{11} \cdot 3$	$-2^{14} \cdot 3^2$	$+2^4$	$+2^8 \cdot 3 \cdot 41$	$-2^4 \cdot 3^3 \cdot 7 \cdot 13$
(6)	$-2^6 \cdot 3^7 \cdot 7$	$+2^{10} \cdot 3 \cdot 5$	$+2^6 \cdot 3^3 \cdot 5 \cdot 29$	$-2^2 \cdot 17$	$-2^7 \cdot 491$	$+2^3 \cdot 3^5 \cdot 5 \cdot 7$
(7)	0	$-2^{12} \cdot 3 \cdot 5$	$-2^6 \cdot 3^6$	$+3 \cdot 7 \cdot 43$	$+2^{10} \cdot 3^4$	0
(8)	-2^{18}	$+2^{11} \cdot 5$	0	$-5 \cdot 47$	$-2^8 \cdot 3^3$	0
(9)	0	$+2^6 \cdot 3 \cdot 5$	-2^{12}	-5^2	$-2^3 \cdot 3^3$	0
(10)	$-2^8 \cdot 3^4$	$+2^6 \cdot 3 \cdot 5$	$+2^4 \cdot 7 \cdot 83$	-2^2	$-2^5 \cdot 101$	$+2 \cdot 3^2 \cdot 5 \cdot 29$
(11)	0	$+2^4 \cdot 3$	0	-1	-2^6	0
(12)	$2^{13} \cdot 3^3$	$+2^9 \cdot 3^2 \cdot 5$	$+2^4 \cdot 191$	$-3^2 \cdot 29$	$-2^6 \cdot 3^3 \cdot 19$	-2^{12}
(13)	0	$-2^8 \cdot 3 \cdot 5$	$+2^{13}$	$+67$	$+2^5 \cdot 3^4$	$+2^{11}$
(14)	0	$-2^8 \cdot 3 \cdot 5$	$-2^5 \cdot 5 \cdot 7 \cdot 13$	$+5^2$	$+2^5 \cdot 5 \cdot 53$	$-2^7 \cdot 3 \cdot 7$
(15)	0	$-2^7 \cdot 3 \cdot 5$	$+2^8 \cdot 3^2$	-19	$-2^5 \cdot 3 \cdot 31$	0
(16)	0	$-2^5 \cdot 3 \cdot 5$	0	$+7$	$+2^2 \cdot 5 \cdot 31$	0

	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	
(1)	$+ 2^4 \cdot 3^7$	$- 2^2 \cdot 3^6$	$+ 3^5$	$- 2^{12} \cdot 3^2$	$+ 2^9 \cdot 3^5$	$- 2^7 \cdot 3^4$	$= 0$
(2)	$- 2^3 \cdot 3^2$	$+ 3^5$	$- 3^3$	$+ 2^9 \cdot 5$	$- 2^5 \cdot 3^3 \cdot 7$	$+ 2^3 \cdot 3^2 \cdot 11$	$= 0$
(3)	$+ 2^3 \cdot 3^4 \cdot 5 \cdot 7$	$- 3^5 \cdot 7$	$+ 2 \cdot 3^2 \cdot 7$	$- 2^{11} \cdot 11$	$+ 2^6 \cdot 3^2 \cdot 143$	$- 2^4 \cdot 3^2 \cdot 43$	$= 0$
(4)	$- 2^5 \cdot 3^4$	$+ 2^3 \cdot 3^4$	$- 2^2 \cdot 3^3$	$+ 2^8 \cdot 5^2$	$- 2^4 \cdot 3^3 \cdot 19$	$+ 2^2 \cdot 3 \cdot \overline{13}^2$	$= 0$
(5)	$+ 2^2 \cdot 3^2 \cdot 61$	$- 2^2 \cdot 3^3$	$+ 2^2$	$- 2^6 \cdot 5^2$	$+ 2^7 \cdot 5 \cdot 13$	$- 2^5 \cdot 13$	$= 0$
(6)	$- 2 \cdot 3^3 \cdot 5 \cdot 17$	$+ 3^3 \cdot 11$	$- 2 \cdot 3^2$	$+ 2^5 \cdot 3 \cdot 43$	$- 2^5 \cdot 3^2 \cdot 59$	$2^2 \cdot 277$	$= 0$
(7)	$+ 2^4 \cdot 3^4 \cdot 5$	$- 2^2 \cdot 3^4 \cdot 5$	$+ 2 \cdot 3^3 \cdot 5$	$- 2^3 \cdot 71$	$+ 2^4 \cdot 3^3 \cdot 53$	$- 2^2 \cdot 3 \cdot 11 \cdot 41$	$= 0$
(8)	0	$+ 2^3 \cdot 3^3$	$- 2^3 \cdot 3^2$	$+ 2^6 \cdot 3^2 \cdot 5$	$- 2^4 \cdot 3^4$	$+ 2^3 \cdot 3^2 \cdot 11$	$= 0$
(9)	0	0	$- 2 \cdot 3$	$+ 2 \cdot 3 \cdot 5 \cdot 7$	0	$+ 2^2 \cdot 3^2$	$= 0$
(10)	$- 241$	$+ 2 \cdot 3^2$	$- 1$	$+ 2^3$	$- 2^2 \cdot 13 \cdot 17$	$+ 2^2 \cdot 17$	$= 0$
(11)	0	0	0	$+ 2 \cdot 3$	0	0	$= 0$
(12)	$- 2^4 \cdot 3^3 \cdot 5$	$+ 2^2 \cdot 3^5$	$- 2^3 \cdot 3^2$	$+ 2^4 \cdot 3 \cdot 7 \cdot 19$	$- 2^3 \cdot 3^4 \cdot 13$	$+ 5 \cdot 353$	$= 0$
(13)	0	$- 2 \cdot 3^3$	$+ 2 \cdot 3^2$	$- 2^3 \cdot 3 \cdot 41$	$+ 2^4 \cdot 3^3$	$- 2^4 \cdot 3 \cdot 5$	$= 0$
(14)	$+ 2^3 \cdot 61$	$- 3^2 \cdot 7$	$+ 2 \cdot 3$	$- 2^3 \cdot 3 \cdot 41$	$+ 2^{11}$	$- 2 \cdot 5^3$	$= 0$
(15)	$- 2^6$	$+ 2 \cdot 3^2$	$- 2^2$	$+ 2^3 \cdot 5 \cdot 11$	$- 2^3 \cdot 67$	$+ 2 \cdot 47$	$= 0$
(16)	0	0	$+ 1$	$- 89$	$+ 2^6$	$- 2^2 \cdot 3$	$= 0$

34. Two invariant forms of degree 18 are already known, A^3 and AP^2 , and since these depend upon two of the forms derived in (7), (8), (9), Art. 16, it is at once suggested that the third form C , which is of degree 18, may also belong to this system. [See (15), Art. 12.]

In fact, it is easily verified that the 16 equations are satisfied by these three forms:

c_1	c_2	c_3	c_4	c_5	c_6
$A^3 = + 2^3 \cdot 3^3$	$+ 3^6$	$+ 2^2 \cdot 3^5$	0	$+ 2 \cdot 3^6$	$+ 2^5 \cdot 3^3 \cdot 5$
$AP^2 = 0$	$- 3^5$	$- 2^3 \cdot 3$	$+ 2^8 \cdot 3^2$	$- 2 \cdot 3^2 \cdot 11$	$+ 2^5 \cdot 3$
$C = - 2^3$	$- 2 \cdot 3^3 \cdot 23$	$- 2 \cdot 17$	$- 2^7 \cdot 151$	$- 2^2 \cdot 73$	$- 2^6$
p_2^9	p_3^6	$p_2^6 p_3^2$	$p_3^2 p_4^3$	$p_2^3 p_3^4$	$p_2^7 p_4$

c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	
$+ 2^7 \cdot 3^2 \cdot 5^2$	$+ 2^9 \cdot 5^3$	0	$+ 2^3 \cdot 3^5 \cdot 5$	$+ 2^5 \cdot 3^4 \cdot 5$	$+ 2^6 \cdot 3^3 \cdot 5^2$	(9)
$- 2^7$	$- 2^9 \cdot 7$	$+ 2^{11} \cdot 5$	$+ 2^3 \cdot 3^3$	$+ 2^4 \cdot 53$	$+ 2^9 \cdot 3^2$	
$+ 2^7$	$- 2^{10} \cdot 13$	$+ 2^{10} \cdot 5 \cdot 11$	$+ 2^4 \cdot 3 \cdot 5^2$	$+ 2^3 \cdot 3^3$	$+ 2^5 \cdot 7 \cdot 11$	
$p_2^5 p_4^2$	$p_2^3 p_4^3$	$p_2 p_4^4$	$p_2 p_3^4 p_4$	$p_2^4 p_3^2 p_4$	$p_2^2 p_3^2 p_4^2$	

35. It follows that an infinity of solutions of the above system of equations exists of the form

$$m_1 A^3 + m_2 AP^2 + m_3 C, \quad (10)$$

where the m 's are arbitrary parameters.

Hence, all determinants of the 12th order in the matrix of the coefficients must vanish and also *all* the first and second minors of these. A third minor, however, is easily found which does not vanish, namely, by depleting the first 7 rows and the 2d, 5th and 10th columns. Thus, no solution exists other than those included in (10).

Therefore, *the only invariant forms of degree 18 are included in the general form*

$$m_1 A^3 + m_2 A P^3 + m_3 C.$$

§5.—THE SYSTEM OF FUNDAMENTAL INVARIANTS.

Invariants of the Linear Groups G_6^{1i} . Arts. 36–38.

36. It has been shown that each of the pencils $1i$ ($i = 2 \dots 5$) is invariant under a dihedron group G_6^{1i} . [Art. 13.]

The known* complete form-systems of these subgroups will now furnish the means of determining the system of fundamental invariants for G_{120} .

These auxiliary systems may be read by the group properties already shown in the configuration Π , Fig. X, Part First.

It will be sufficient to deduce the system for one pencil, say at the vertex 13.

The three conics $13 \cdot 24, 13 \cdot 25, 13 \cdot 45$

are permuted under the dihedron

$$G_6^{13} \sim \{245\} \text{ all.}$$

Hence, this group must also permute among themselves those directions in the pencil 13 which are given by the corresponding tangents to these conics at that point,

$$\left. \begin{aligned} z_1 - 2z_2 &= 0, \\ 2z_1 - z_2 &= 0, \\ z_1 + z_2 &= 0. \end{aligned} \right\} \quad (1)$$

The product of these tangential quantities is, therefore, one of the cubic invariant forms under G_6^{13} ,

$$f_1 = -2z_1^3 + 3(z_1^2 z_2 + z_1 z_2^2) - 2z_2^3. \quad (2)$$

* Klein, "Ikosaeder," Kapitel I, §9.

The other cubic invariant is the product of the three sides of the quadrangle Π_1 which pass through the point 13,

$$f_2 = z_1^2 z_2 - z_1 z_2^2. \quad (3)$$

37. In order to find the quadratic invariant under G_6^{13} , we take the invariant conic belonging to the fundamental system of $G_{24}^{(1)}$ [Art. 14],

$$p_2 \equiv -3\Sigma z_1^2 + 2\Sigma z_1 z_2 = 0 \quad (4)$$

and find the tangents to it from the point 13, namely [Art. 46, Part First],

$$\left. \begin{aligned} \omega z_1 + z_2 &= 0, \\ \omega^2 z_1 + z_2 &= 0. \end{aligned} \right\} \quad (5)$$

The product of these tangential quantities is, then, the quadratic invariant form of G_6^{13} ,

$$f_3 = z_1^2 - z_1 z_2 + z_2^2. \quad (6)$$

This follows since, under G_6^{13} , *the point 13 is fixed and the conic is fixed*, so that the tangents must be either fixed or permuted, making their product an invariant. Since there is *only one quadratic invariant* under G_6^{13} , it must be f_3 thus determined.

In like manner the systems may be found for the other pencils and their groups.

38. It will be seen that the above forms f_1, f_2, f_3 correspond to Klein's forms F_1, F_2, F_3 in the following manner:

$$\left. \begin{array}{ccc} F_1 & F_2^2 & F_3^3 \\ f_1 & -3^3 f_2^2 & 2^2 f_3^3 \end{array} \right\} \quad (7)$$

so that the identity* holds

$$F_1^2 - F_2^2 - F_3^3 \equiv f_1^2 + 3^3 f_2^2 - 2^2 f_3^3 = 0. \quad (8)$$

For the point 13 we have

$$\left. \begin{aligned} f_1^2 &= 4(z_1^6 + z_2^6) - 12(z_1^5 z_2 + z_1 z_2^5) - 3(z_1^4 z_2^2 + z_1^2 z_2^4) + 26z_1^3 z_2^3, \\ f_2^2 &= + (z_1^4 z_2^2 + z_1^2 z_2^4) - 2z_1^3 z_2^3, \\ f_3^3 &= (z_1^6 + z_2^6) - 3(z_1^5 z_2 + z_1 z_2^5) + 6(z_1^4 z_2^2 + z_1^2 z_2^4) - 7z_1^3 z_2^3. \end{aligned} \right\} \quad (9)$$

From (9) we see at once that (8) holds.

* Klein, "Ikosaeder," p. 49.

The only remaining type-form of the 6th degree under G_6^{13} is

$$f_1 f_2 = -2(z_1^5 z_2 - z_1 z_2^5) + 5(z_1^4 z_2^2 - z_1^2 z_2^4),^* \quad (10)$$

which alone is non-symmetric in z_1, z_2 .

Relation of A, P^2, C to f_1, f_2, f_3 . Arts. 39-47.

39. It has been seen by theorems I, II [Arts. 23, 24] that every invariant form under G_{120} may be reduced to

$$R_{6n} = P^{2\mu} \cdot R_{6(n-2\mu)},$$

and that it has at each of the critical points a multiple point of order $2(n + \mu)$.

For the forms already considered the parameters, n and μ have the following special values :

	n	μ	$2(n + \mu)$
A	1	0	2
A^2	2	0	4
A^3	3	0	6
P^2	2	1	6
AP^2	3	2	10
C	3	0	6

(1)

40. Consider the sextic curve

$$A = 0. \quad (2)$$

* See Art. 46.

The product of its tangential quantics at 13, one of its double points, is the binary quadratic invariant for that point. For

$$A = 0$$

is fixed, and the point 13 is fixed under the linear group G_6^{13} . Therefore, the two tangents are either fixed or permuted, and their product is the quadratic invariant f_3 , since only one such form exists under G_6^{13} .

In this sense the ternary form A is said to correspond to the binary form f_3 , and since A is the only ternary invariant form of the sixth degree, the correspondence is one to one.

If, however, the second polar of (2) with respect to the vertex 13 be formed, f_3 is found to carry a numerical factor, thus

$$A \sim 2f_3. \quad (3)$$

41. The correspondence is very different in the case of the degenerate curve.

$$P^2 = 0, \quad (4)$$

which represents the six sides, each taken twice. The product of these at 13 is exactly f_2^2 , so that

$$P^2 \sim f_2^2. \quad (5)$$

Whereas, the other 12th degree form A^2 gives the correspondence

$$A^2 \sim 2^2 f_3^2. \quad (6)$$

This illustrates theorem III [Art. 24], *showing how the factor*

$$P^{2\mu} \text{ in the form } R_{6n}, \quad (\mu \neq 0)$$

raises the order of the multiple point by 2μ on account of the degeneracy of the curve (4).

For this reason the correspondence (5) is *not one to one*, since f_2^2 may go equally well with some form of degree 18. See (10), Art. 43.

42. In order to find the binary correspondent to the curve

$$C = 0, \quad (7)$$

we form the 6th polar of (7) with reference to the point

$$13; 0:0:1. \quad (8)$$

For this purpose the following remarks are useful :

$$(a). \quad \frac{\partial p}{\partial z_i} = 1, \quad \frac{\partial q}{\partial z_i} = z_j + z_k, \quad \frac{\partial r}{\partial z_i} = z_j z_k, \\ (i, j, k = 1, 2, 3).$$

(b). At the point (8),

$$p = 1, \quad q = 0, \quad r = 0.$$

(c). Neither the factor p^α nor any of its derivatives below the $(\alpha + 1)^{\text{st}}$ can cause any term to vanish.

(d). The factor q^β or any of its derivatives up to and including the β^{th} , save one,

$$\frac{\partial^\beta q^\beta}{\partial z_i^\beta}, \quad (i = 1, 2)$$

will cause all terms to vanish in which it may occur.

(e). The factor r^γ or any of its derivatives up to and including the $2\gamma^{\text{th}}$, save one,

$$\frac{\partial^{2\gamma} r^\gamma}{\partial z_1^\gamma \partial z_2^\gamma}$$

will cause all terms to vanish in which it may occur.

(f). Hence, the λ^{th} derivative of a term, $p^\alpha q^\beta r^\gamma$, will vanish for the point, $0:0:1$, unless

$$\beta + 2\gamma \leq \lambda.$$

43. Applying these principles to the curve (9), Art. 16, we see at once that every term has

$$\beta + 2\gamma \leq 5.$$

Hence, the first five polars must vanish. But for $\lambda = 6$ there are two terms whose sixth derivatives do not vanish,

$$- 2^2 p^3 r^3 \text{ and } p^8 q^2 r^2.$$

Determining from these the sixth polar, the product of the tangential quantics at the sextuple point 13 of (7) is found to be

$$z_1^4 z_2^2 - 2z_1^3 z_2^3 + z_1^2 z_2^4. \quad (9)$$

This is a perfect square, and is precisely the dihedron form f_2^2 . [(9), Art. 38.]

Therefore, *the curve (7) has three cusps at the point 13*, and hence the same also at the other three critical points.

Thus we have $C \sim f_2^2$. (10)

This may be called the *normal* correspondence between f_2^2 and a ternary form of degree 18, while that between f_2^2 and P^2 of degree 12 *exists only because* (4) breaks up into straight lines through the critical points.

44. It remains to discover the ternary form to which f_1^2 corresponds. From the identity (8), Art. 38, we derive

$$2f_1^2 = 2^3f_3^3 - 2 \cdot 3^3f_2^2. \quad (11)$$

From (3) and (10),

$$A^3 - 2 \cdot 3^3C \sim 2^3f_3^3 - 2 \cdot 3^3f_2^2. \quad (12)$$

Calling the left of (12) C' , we have

$$C' \sim 2f_1^2. \quad (13)$$

45. The above results may be made more general as follows :

THEOREM IV.*

If the ternary forms α, β correspond to the binary forms a, b of degree λ, μ , then the product

$$\alpha\beta \sim ab$$

and the sum

$$\alpha + \beta \sim a, b \text{ or } a + b,$$

according as

$$\lambda < \mu, > \mu \text{ or } = \mu.$$

The proof follows directly from the principles of higher plane curves, since the correspondence in question is determined by finding the *lowest non-vanishing polar of each curve with respect to the given point*.

By this theorem a more general form than C corresponds to f_2^2 , namely,

$$C + \delta AP^2 \sim f_2^2, \quad (14)$$

where δ is an arbitrary constant.

Here the binary correspondent of C , which is of *lower degree* than that of AP^2 [Art. 39, (1)], *prevails also* for the composite form.

* The chief theorems are numbered consecutively. See Arts. 22, 23, 24.

Likewise C' [(12), (13)] may be generalized, for if we put

$$A^3 - 2 \cdot 3^3 C + \delta A P^2 = C'',$$

we have

$$C'' \sim 2f_1^2. \quad (15)$$

46. THEOREM V.

No invariant form under G_{120} can correspond to any binary form which involves odd powers of the cubic forms f_1, f_2 .

Proof: If an odd power of either f_1 or f_2 alone is involved, then the form is of odd degree. This is impossible, since the degree of the *binary* form must equal the order of multiplicity of the point in question, which is always $2(n + \mu)$ according to theorem III, Art. 24.

If an odd power of the product only, $f_1 f_2$, is involved, the form is then of even degree, but is *non-symmetric* in z_1, z_2 [(10), Art. 38]. This is impossible, since the *ternary* forms are symmetric functions of z_1, z_2, z_3 . [Art. 20]. Hence, all their polars with respect to the coordinate vertex, $0:0:1$, through which all the invariant curves pass [Art. 24], are *symmetric* functions in z_1, z_2 , and similar statements hold with reference to the other three fundamental points.

Therefore, all invariant forms under G_{120} have as their binary correspondents at the point* 13, rational, integral functions of

$$f_1^2, f_2^2, f_3.$$

47. The complete enumeration of binary correspondents for all ternary forms of degree 6, 12 and 18 may now be given as follows:

$$m_1 A \sim 2m_1 f_3,$$

$$m_1 A^2 + m_2 P^2 \sim \begin{cases} 2^2 m_1 f_3^2, & \text{if } m_1 \neq 0, \\ m_2 f_2^2, & \text{if } m_1 = 0, m_2 \neq 0. \end{cases}$$

$$m_1 A^3 + m_2 C + m_3 A P^2 \sim \begin{cases} m_2 f_2^2, & \text{if } m_1 = 0, m_2 \neq 0; \\ 2^3 m_1 f_3^3, & \text{if } m_1 \neq 0, m_2 = 0; \\ 2m_3 f_2^2 f_3^2, & \text{if } m_1 = m_2 = 0, m_3 \neq 0; \\ 2m_1 f_1^2, & \text{if } m_2 = -2 \cdot 3^3 m_1, \neq 0; \\ 2^3 m_1 f_3^3 + m_2 f_2^2, & \text{if } m_2 \neq -2 \cdot 3^3 m_1 \neq 0. \end{cases}$$

*The foregoing investigation, which has been given with respect to the point 13, applies equally well to all four critical points, since these are conjugate under $G_{120}^{(1)}$.

The simplest invariant forms independent of P^2 corresponding directly to the fundamental even binary forms are

$$\left\{ \begin{array}{l} \frac{1}{2} A \sim f_3, \\ C \sim f_2^2, \\ \frac{1}{2} C' \sim f_1^2. \end{array} \right\} \quad (16)$$

THE GENERAL REDUCTION THEOREM. Arts. 48–52.

48. *Definition.* An invariant form is said to be *reduced* if it does not contain P^2 as a factor.

LEMMA.

Two reduced invariant forms of the same degree, which have the same binary correspondent at one of the vertices, can differ only in such terms as contain P^2 as a factor.

Proof: Let R and R' be two reduced ternary forms, each of degree $6n$, having the same tangential quantic at the point 13. We are to prove

$$R - R' = P^{2\mu} \cdot R''_{6(n-2\mu)}, \quad (1)$$

where R'' contains no factor of P and is zero if $\mu = 0$.

Since R and R' are *reduced* forms, the common tangential quantic at the point $0:0:1$, is of degree $2n$ [Art. 24], and involves only z_1 and z_2 [Art. 46]. Hence, by principles of higher plane curves,

- (a). R and R' each contain no terms of degree less than $2n$ in z_1 and z_2 .
- (b). They are *identical* in the terms of degree $2n$ in z_1 and z_2 .
- (c). And, therefore, they can differ only in terms of degree *higher* than $2n$ in z_1 and z_2 .

Now, $R - R'$ is an invariant form of degree $6n$, since R and R' are such by hypothesis. Then $R - R'$ must have at 13 a tangential quantic of degree $2(n + \mu)$. [Art. 24.]

Two cases now arise;

(I). $\mu = 0$. The tangential quantic is then of degree $2n$ which, by (c), is impossible, since $R - R'$ contains no terms of degree $\leq 2n$ in z_1 and z_2 . Hence,

$$R - R' = 0. \quad (2)$$

(II). $\mu \neq 0$. This means that $R - R'$ actually contains $P^{2\mu}$ as a factor, in

which case alone [Art. 41] can a form have a tangential quantic of degree higher than $2n$. Hence, the conclusion (I) is established.

Corollary.

49. If one of two ternary forms of the same degree has the factor $P^{2\mu}$ ($\mu \geq 1$), while the other does not, the degree of the binary correspondent of the former is greater by 2μ than that of the latter.

THEOREM VI.

50. The most general invariant form under G_{120} is a rational integral function of the forms A , P^2 , C .

Proof: Let R_1 be the most general invariant form of degree $6n$, and let Q_1 be the resulting *reduced* form of degree $6(n - 2\mu_1)$.

Suppose the binary correspondent of Q_1 is given by

$$Q_1 \sim G_1(f_3, f_2^2, f_1^2), \quad (1)$$

where G_1 is a rational integral function of the *even* binary forms of degree $2(n - 2\mu_1)$ in z_1, z_2 [Arts. 24, 46.]

Now, a known invariant form of degree $6(n - 2\mu_1)$ in z_1, z_2, z_3 can be constructed, having G_1 for its binary correspondent,

$$S_1 \equiv G_1(\tfrac{1}{2}A, C, \tfrac{1}{2}C'). \quad (2)$$

For the arguments in (1) are weighted 2, 6, 6 respectively in the variables, and they are the binary forms corresponding *directly* to the arguments in (2), which are weighted 6, 18, 18 respectively. See (16), Art. 47.

Therefore, Q_1 and S_1 are two *reduced* forms having the same binary correspondent, which, by the lemma [Art. 48] are then either identical or differ only in terms involving P^2 .

Thus, either
$$Q_1 - S_1 \equiv 0, \quad (3)$$

in which case the theorem is proved, or

$$Q_1 - S_1 = R_2, \quad (4)$$

where R_2 is divisible by $P^{2\mu_2}$ ($\mu_2 \geq 1$).

Call the quotient Q_2 of degree $6(n - 2\mu_1 - 2\mu_2)$ and suppose

$$Q_2 \sim G_2(f_3, f_2^2, f_1^2). \quad (5)$$

Then, as before, set up the known form

$$S_2 \equiv G_2(\tfrac{1}{2}A, C, \tfrac{1}{2}C') \quad (6)$$

of degree $6(n - 2\mu_1 - 2\mu_2)$ and having the same binary correspondent as Q_2 .

Hence, either
$$Q_2 - S_2 = 0, \quad (7)$$

in which case

$$R_2 = P^{2\mu_2} G_2$$

and

$$\begin{aligned} R_1 &= P^{2(\mu_2 + \mu_1)} G_2 + P^{2\mu_1} G_1 \\ &= \phi(P^2, A, C), \end{aligned} \quad (8)$$

or else

$$Q_2 - S_2 = R_3, \quad (9)$$

where R_3 is divisible by $P^{2\mu_3}$ ($\mu_3 \geq 1$).

If this process be continued, we must reach, after a finite number of steps κ , a form independent of P^2 , since the finite degree $6n$ is successively reduced by multiples of 6, the divisor being always a power of P^2 .

Hence, *after* the κ^{th} reduction, we shall have

$$Q_\kappa - S_\kappa = 0 \quad (10)$$

and

$$Q_\kappa = S_\kappa = G_\kappa(\tfrac{1}{2}A, C, \tfrac{1}{2}C'), \quad (11)$$

where G_κ is a function of degree $6[n - 2(\mu_1 + \mu_2 + \dots + \mu_\kappa)]$ in z_1, z_2, z_3 . So that

$$R_\kappa = P^{2\mu_\kappa} G_\kappa. \quad (12)$$

But

$$Q_{\kappa-1} = R_\kappa + G_{\kappa-1} = P^{2\mu_\kappa} G_\kappa + G_{\kappa-1}.$$

Hence

$$R_{\kappa-1} = P^{2(\mu_\kappa + \mu_{\kappa-1})} G_\kappa + P^{2\mu_{\kappa-1}} G_{\kappa-1}.$$

Likewise,

$$\begin{aligned} R_{\kappa-2} &= P^{2(\mu_\kappa + \mu_{\kappa-1} + \mu_{\kappa-2})} G_\kappa + P^{2(\mu_{\kappa-1} + \mu_{\kappa-2})} G_{\kappa-1} + P^{2\mu_{\kappa-2}} G_{\kappa-2}, \\ R_{\kappa-3} &= P^{2(\mu_\kappa + \mu_{\kappa-1} + \mu_{\kappa-2} + \mu_{\kappa-3})} G_\kappa + P^{2(\mu_{\kappa-1} + \mu_{\kappa-2} + \mu_{\kappa-3})} G_{\kappa-1} \\ &\quad + P^{2(\mu_{\kappa-2} + \mu_{\kappa-3})} G_{\kappa-2} + P^{2\mu_{\kappa-3}} G_{\kappa-3}, \\ &\dots\dots\dots \\ R_1 &= P^{2(\mu_\kappa + \mu_{\kappa-1} + \dots + \mu_1)} G_\kappa + P^{2(\mu_{\kappa-1} + \mu_{\kappa-2} + \dots + \mu_1)} G_{\kappa-1} + \dots + P^{2\mu_1} G_1. \end{aligned}$$

Therefore, we have

$$\begin{aligned} R_1 &= \phi_1(P^2, G_1, G_2, \dots, G_\kappa) \\ &= \phi_2(P^2, A, C, C') \\ &= \phi(P^2, A, C), \end{aligned} \tag{13}$$

where the ϕ 's are rational, integral functions.

51. Hence, *every absolute invariant form under G_{120} , throwing off the required factor in z_1, z_2, z_3 , is a rational integral function of the fundamental forms A, P^2, C .*

Out of these forms must be constructed the invariant fractions, which return absolutely to themselves after canceling the common factor in numerator and denominator thrown off under any quadratic transformation of the group.

Since A is the only form of degree 6, there is no fraction of that order. The simplest fractions of degree 12, 18 and 24 respectively are

$$\frac{A^2}{P^2}; \frac{A^3}{C}, \frac{AP^2}{C}; \frac{A^4}{P^4}, \frac{AC}{P^4}.$$

52. It was shown [Art. 17] that the form P is the only *fundamental relative invariant form under $G_{24}^{(1)}$* , and that it throws off the factor (-1) . It also throws off (-1) under the generator T' in addition to the factor r^2 in the z 's required by the theorem of Art. 22.

It is, therefore, the *fundamental relative invariant form under G_{120}* , and all other relative invariants are formed from the product

$$P^{2\kappa+1} \cdot \phi(A, P^2, C).$$

However, P is an absolute invariant under the alternating group G_{60} , and the complete form-system for this subgroup is, therefore,

$$A, P, C.$$

There is then for G_{60} an absolute invariant fraction of degree 6,

$$\frac{A}{P}.$$

There is no form of degree 12 in addition to those given in Art. 51

For the degrees 18 and 24 the following non-reducible forms for G_{60} are to be added :

$$\frac{A^3}{P^3}, \quad \frac{C}{P^3}, \quad \frac{A^2P}{C}; \quad \frac{A^4}{PC}.$$

It will be seen that these fractions are all of type (III) or (IV), Art. 19, in which forms all relative invariant fractions under G_{120} must occur.

THE UNIVERSITY OF CHICAGO, Dec. 1, 1900.

NOTE.—In Part First, the last foot-note to Art. 1 should read, pp. 279–291, and the last foot-note to Art. 3 should read, p. 283. The heading above Art. 17 should read, Arts. 17–20.

